

# Transformaciones lineales del plano

16 de octubre de 2026

# Espacio $\mathbb{R}^2$

- ▶ Vectores:  $v = \begin{pmatrix} x \\ y \end{pmatrix}$ ,  $x, y \in \mathbb{R}$ .
- ▶ Vector nulo:  $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ .
- ▶ Suma:  $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$ .
- ▶ Producto por escalar:  $\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}$ .

# Base canónica de $\mathbb{R}^2$

►  $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

► Independencia lineal:

$$\alpha e_1 + \beta e_2 = 0 \implies \alpha = \beta = 0.$$

► Cada vector es una combinación lineal de la base, de manera única:

$$\begin{pmatrix} x \\ y \end{pmatrix} = x e_1 + y e_2.$$

# Matriz de aplicación lineal $\mathbb{R}^2 \rightarrow \mathbb{R}^2$

- ▶ Aplicación  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ .
- ▶ Linealidad:  $f(u + v) = f(u) + f(v)$ ,  $f(\alpha v) = \alpha f(v)$ .
- ▶ Matriz de  $2 \times 2$ :

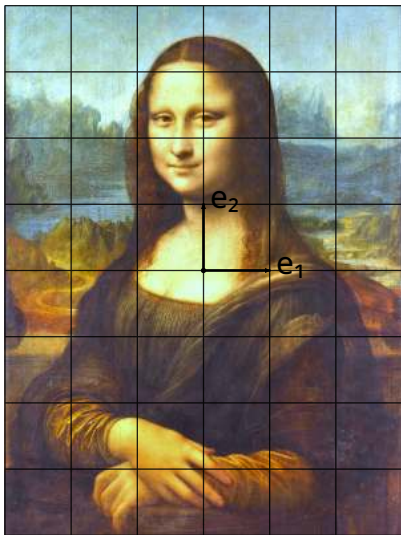
$$f(e_1) = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, \quad f(e_2) = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

- ▶  $f(v) = Av$ :

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix}.$$

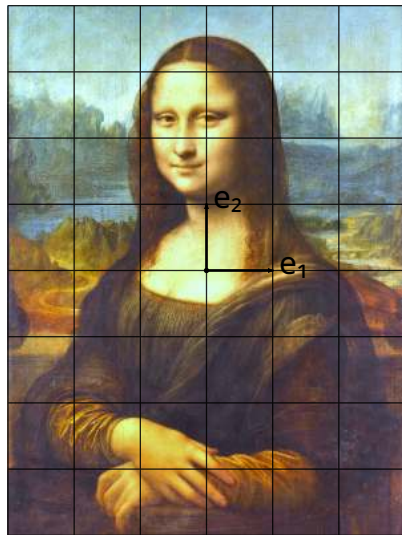
# Aplicación identidad

- ▶ Deja vectores intactos.
- ▶  $id: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $id(v) = v$ .
- ▶ Matriz identidad  $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .



$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

→



# Composición de aplicaciones

- ▶  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ .
- ▶  $A = \text{matriz de } f$ ,  $B = \text{matriz de } g$ .
- ▶ Composición  $g \circ f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ :

$$(g \circ f)(v) = g(f(v)).$$

- ▶ La matriz de  $g \circ f$  es  $BA$ :

$$\underbrace{B \left( \underbrace{A \begin{pmatrix} x \\ y \end{pmatrix}}_{f(v)} \right)}_{g(f(v))} = (BA) \begin{pmatrix} x \\ y \end{pmatrix}.$$

# Aplicaciones y matrices inversas

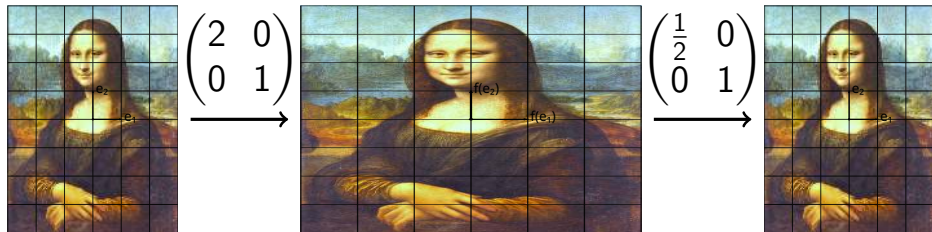
►  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ .

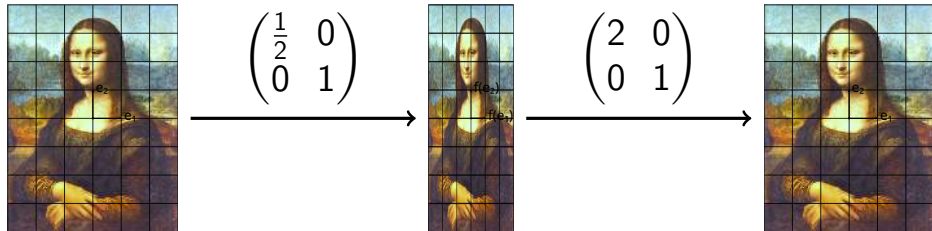
► Aplicación inversa:  $f^{-1}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  tal que

$$f^{-1} \circ f = f \circ f^{-1} = id.$$

► Matriz de  $f^{-1}$  es  $A^{-1}$ :

$$A^{-1}A = AA^{-1} = I.$$





# Rotaciones

- ▶ Rotación por  $\phi$  en sentido antihorario:  $\begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$ .
- ▶  $\begin{pmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} =$   
 $\begin{pmatrix} \cos(\phi + \psi) & -\sin(\phi + \psi) \\ \sin(\phi + \psi) & \cos(\phi + \psi) \end{pmatrix}$ .
- ▶  $\begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}^{-1} = \begin{pmatrix} \cos(-\phi) & -\sin(-\phi) \\ \sin(-\phi) & \cos(-\phi) \end{pmatrix}$ .

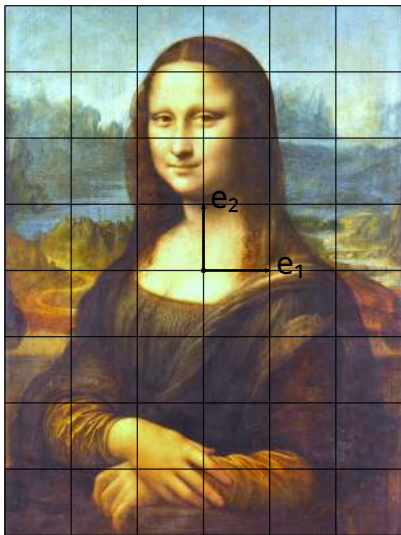
# Casos particulares

$$\begin{pmatrix} \cos 0^\circ & -\operatorname{sen} 0^\circ \\ \operatorname{sen} 0^\circ & \cos 0^\circ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

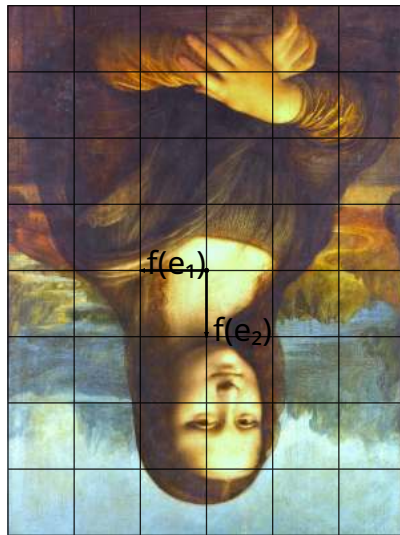
$$\begin{pmatrix} \cos 180^\circ & -\operatorname{sen} 180^\circ \\ \operatorname{sen} 180^\circ & \cos 180^\circ \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

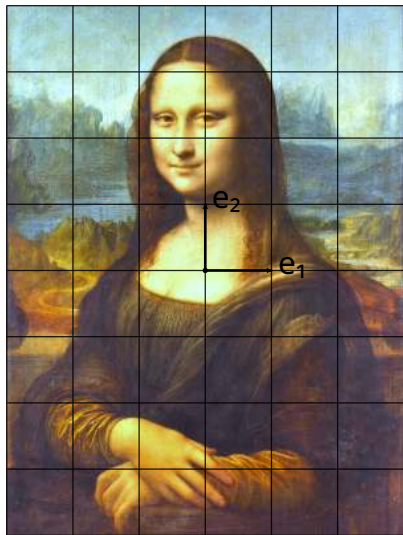
$$\begin{pmatrix} \cos 90^\circ & -\operatorname{sen} 90^\circ \\ \operatorname{sen} 90^\circ & \cos 90^\circ \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \cos(-90^\circ) & -\operatorname{sen}(-90^\circ) \\ \operatorname{sen}(-90^\circ) & \cos(-90^\circ) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$



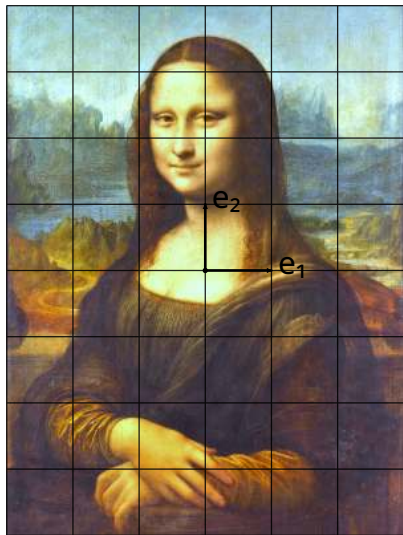
$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow$$





$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightarrow$$

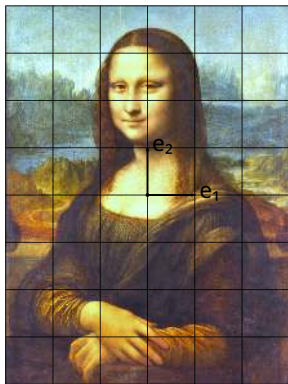




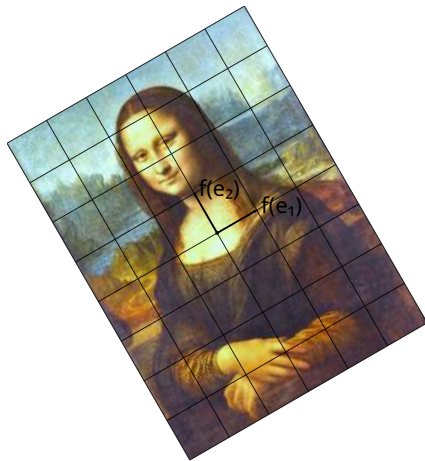
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

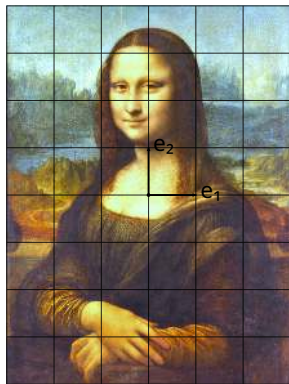
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$$\begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix}$$

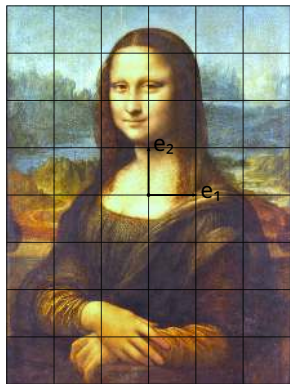




$$\begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix}$$

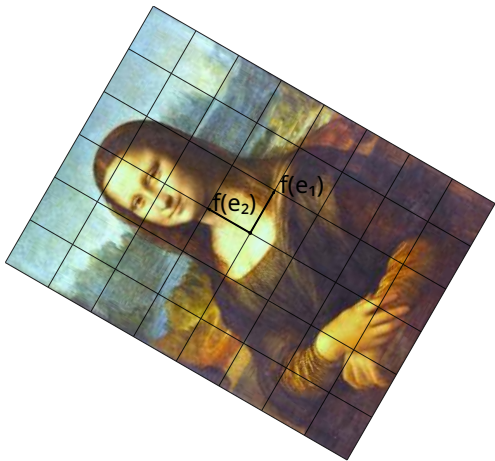
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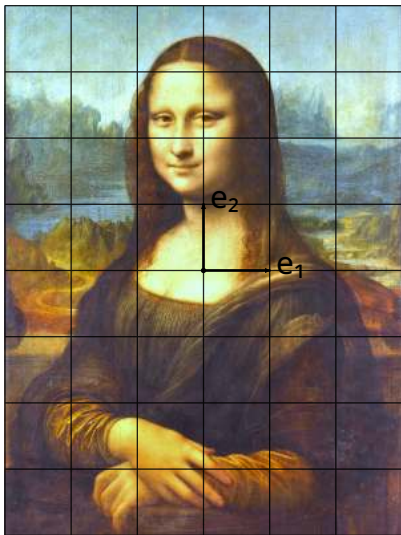
$$\begin{pmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{pmatrix}$$

→

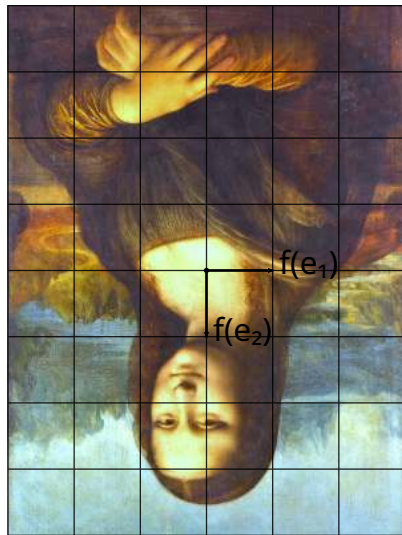


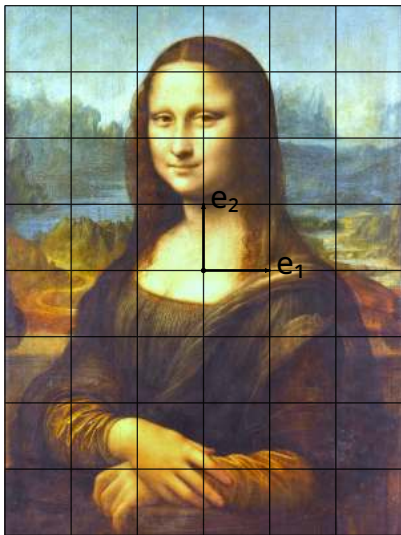
# Reflexiones

- ▶ Respecto al eje  $x$ :  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .
- ▶ Respecto al eje  $y$ :  $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ .
- ▶ Respecto a  $y = x$  (permutación de coordenadas):  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .
- ▶  $f \circ f = id, f^{-1} = f$ .



$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow$$

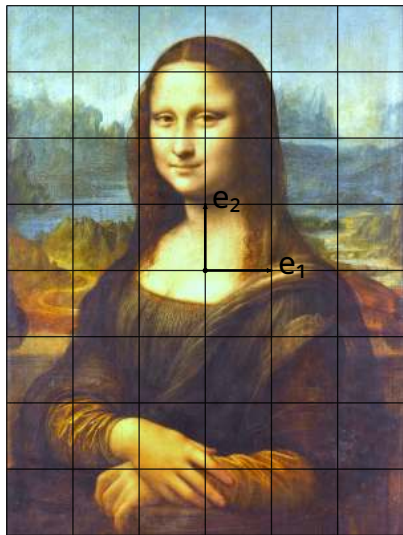




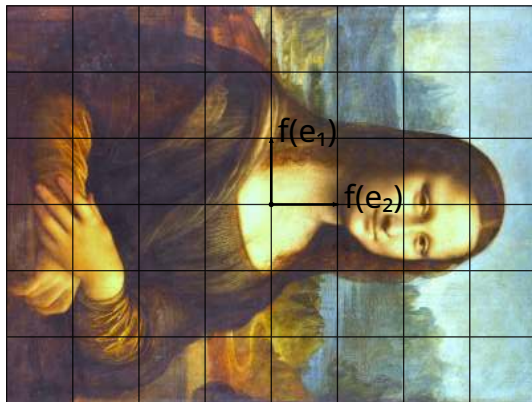
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

→





$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow$$

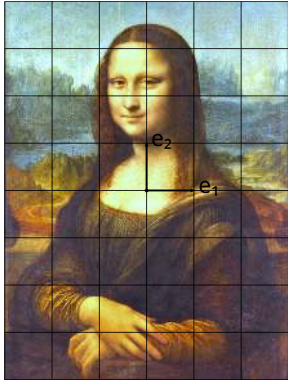




$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

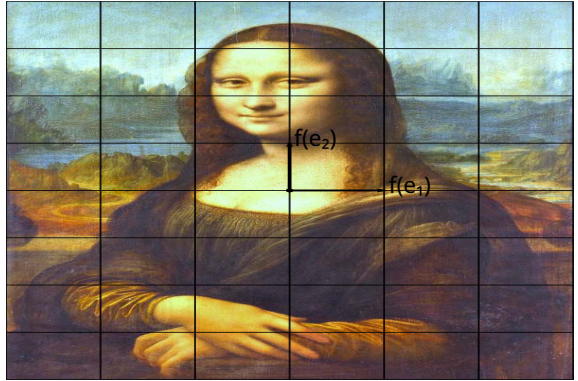
# Escalamiento

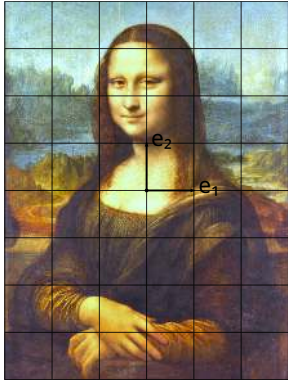
- ▶ Horizontal (a lo largo de  $x$ ):  $\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$ .
- ▶ Vertical (a lo largo de  $y$ ):  $\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$ .
- ▶ En ambas direcciones:  $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ .
- ▶  $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}$ .



$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

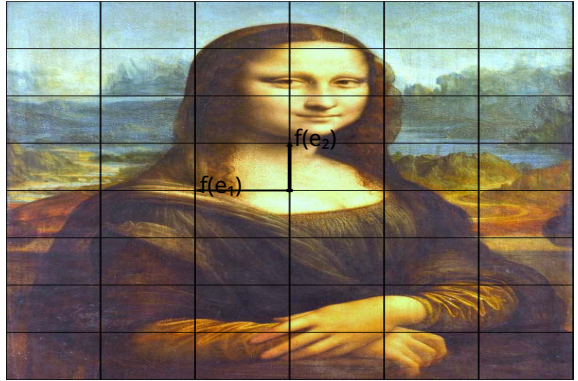
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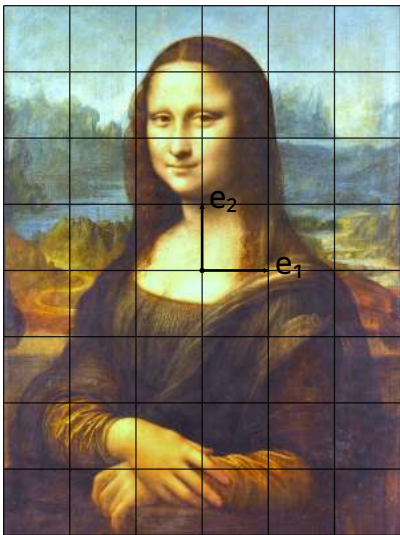




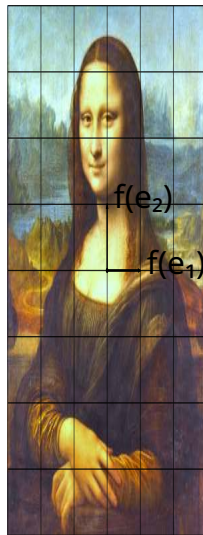
$$\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$$

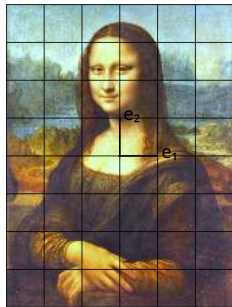
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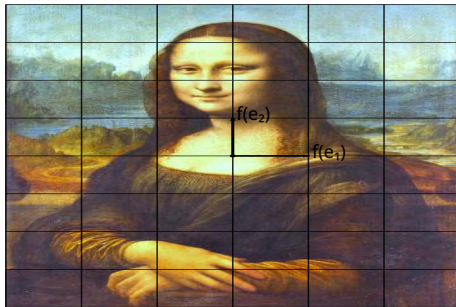


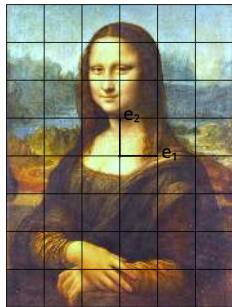
$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$





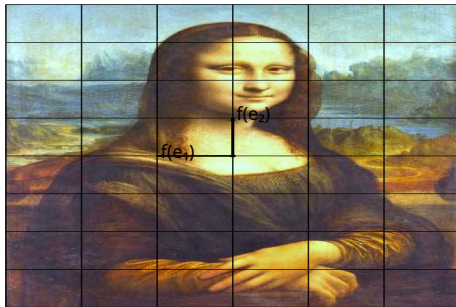
$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$





$$\begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}$$

→





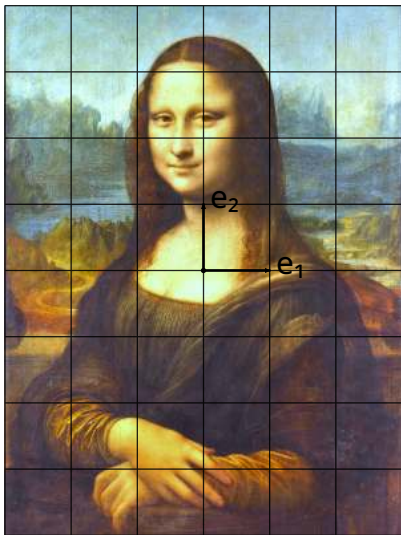
$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$





$$\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$





$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

→





$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$





$$\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$$

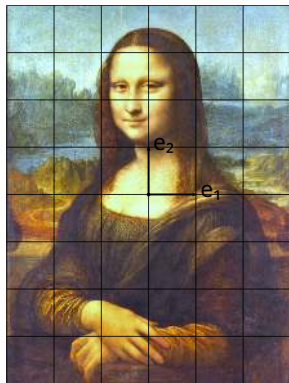


# Transvección (shear)

► Horizontal:  $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ .

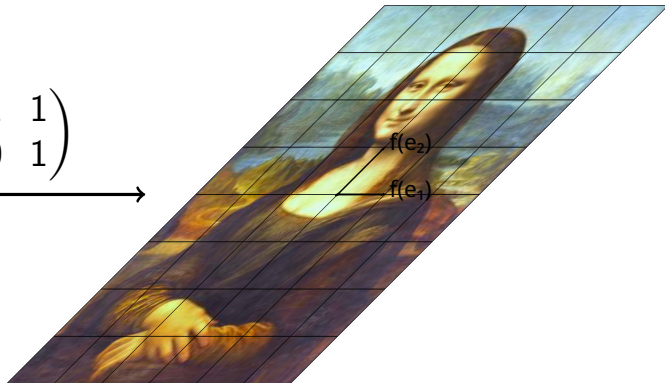
► Vertical:  $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ .

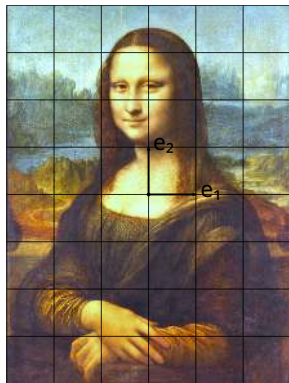
► Inversas:  $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}$ .



$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

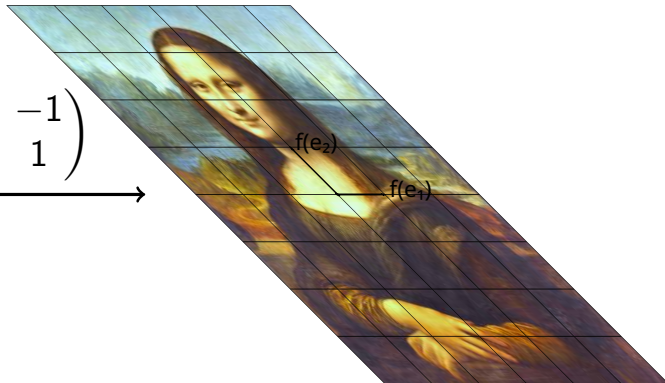
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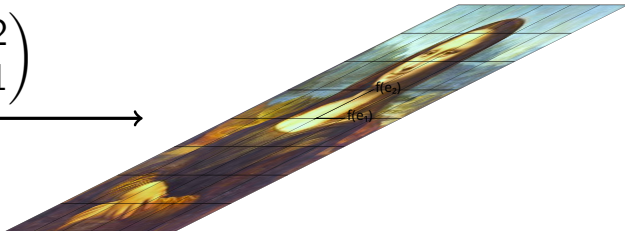
$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

→





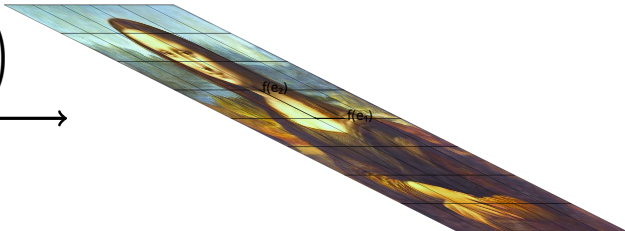
$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

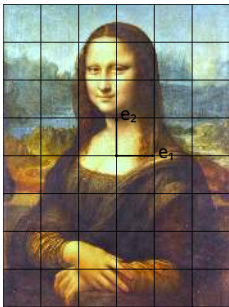




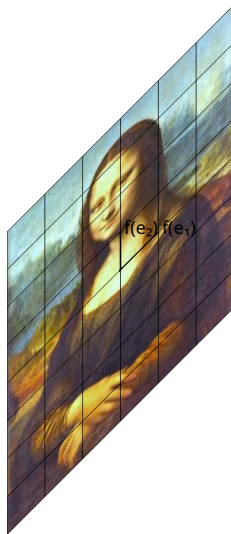
$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

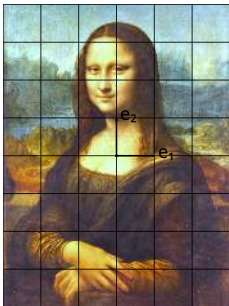
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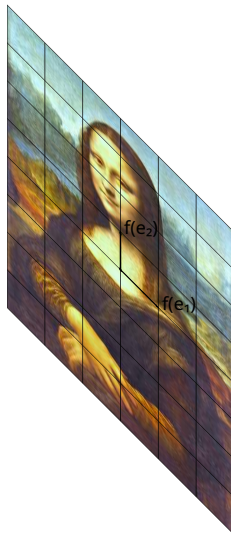


$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$



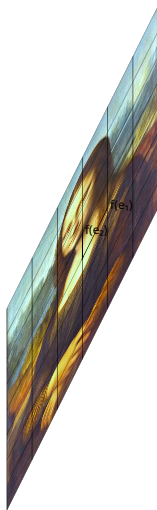


$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$



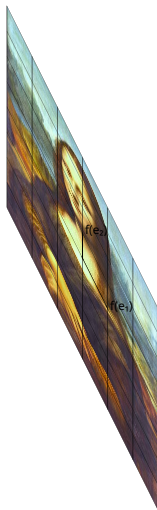


$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$



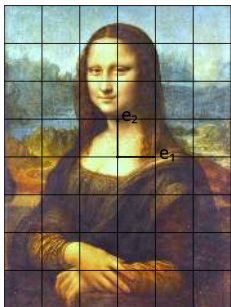


$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

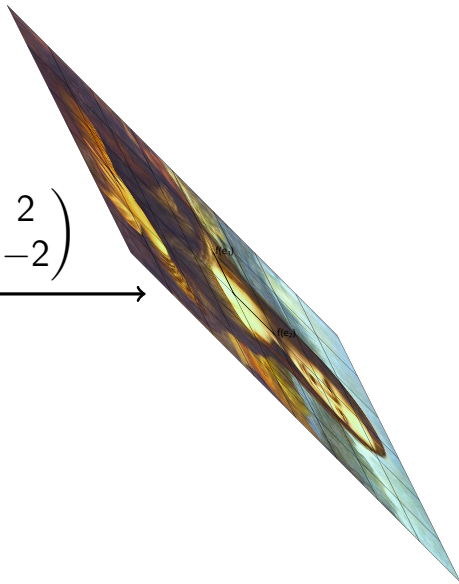


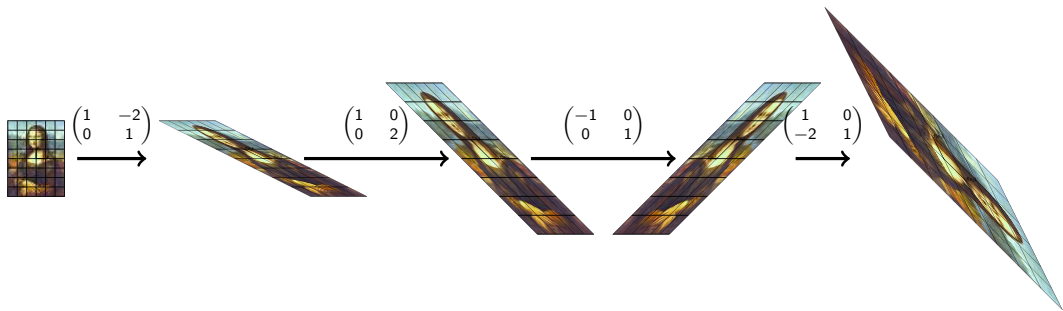
Toda matriz invertible es un producto de matrices elementales

$$\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix};$$
$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix};$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$



$$\begin{pmatrix} -1 & 2 \\ 2 & -2 \end{pmatrix}$$





$$\begin{pmatrix} -1 & 2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

# Significado del determinante

- ▶  $\det A = \text{área del paralelogramo con lados } f(e_1), f(e_2).$
- ▶  $\det A = 0 \iff \text{paralelogramo se aplastó} \iff f \text{ no invertible.}$



$$\begin{pmatrix} 1 & -1 \\ -2 & 4 \end{pmatrix}$$

→

