Zeta-values of one-dimensional arithmetic schemes at strictly negative integers

Alexey Beshenov

Abstract

Let X be an arithmetic scheme (i.e., separated, of finite type over $\operatorname{Spec} \mathbb{Z}$) of Krull dimension 1. For the associated zeta function $\zeta(X, s)$, we write down a formula for the special value at $s = n < 0$ in terms of the étale motivic cohomology of X and a regulator. We prove it in the case when for each generic point $\eta \in X$ with $\operatorname{char} \kappa(\eta) = 0$, the extension $\kappa(\eta)/\mathbb{Q}$ is abelian. We conjecture that the formula holds for any one-dimensional arithmetic scheme.

This is a consequence of the Weil-étale formalism developed by the author in [2] and [3], following the work of Flach and Morin [8]. We also calculate the Weil-étale cohomology of one-dimensional arithmetic schemes and show that our special value formula is a particular case of the main conjecture from [3].

1 Introduction

Let X be an **arithmetic scheme**, by which we mean in this text that it is separated and of finite type over $\operatorname{Spec} \mathbb{Z}$. The **zeta function** associated to X (see, e.g. [36]) is given by

$$\zeta(X, s) := \prod_{\substack{x \in X \\ \text{closed pt.}}} \frac{1}{1 - N(x)^{-s}},$$

where the norm of a closed point $x \in X$ is the size of the corresponding residue field:

$$N(x) := |\kappa(x)| := |\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}|$$

The above product converges for $\operatorname{Re} s > \dim X$ and is supposed to have a meromorphic continuation to the whole complex plane. Although the latter is a wide-open conjecture in general, it is well-known for one-dimensional schemes, which is the case of interest in this article.

If $\zeta(X, s)$ admits a meromorphic continuation around $s = n$, we denote by

$$d_n := \operatorname{ord}_{s=n} \zeta(X, s) \tag{1}$$

the **vanishing order** of $\zeta(X, s)$ at $s = n$. The corresponding **special value** of $\zeta(X, s)$ at $s = n$ is defined as the leading nonzero coefficient of the Taylor expansion:

$$\zeta^*(X, n) := \lim_{s \to n} (s - n)^{-d_n} \, \zeta(X, s).$$

Since the 19th century, many formulas (both conjectural and unconditional) have been proposed to interpret the numbers $\zeta^*(X, n)$ in terms of geometric and algebraic invariants attached to X. A primordial example is Dirichlet's **analytic class number formula**. For a number field $F/\mathbb{Q}$, we denote by $\mathcal{O}_F$ the corresponding ring of integers. Then

$$\zeta_F(s) := \zeta(\operatorname{Spec} \mathcal{O}_F, s)$$

is the **Dedekind zeta function** attached to F. From the well-known functional equation for $\zeta_F(s)$, it is easy to see that it has a zero at $s = 0$ of order $r_1 + r_2 - 1$, where r_1 (resp. $2r_2$) is the number of real embeddings $F \hookrightarrow \mathbb{R}$ (resp. complex embeddings $F \hookrightarrow \mathbb{C}$). The corresponding special value at $s = 0$ is given by

$$\zeta_F^*(0) = -\frac{h_F}{\omega_F} \, R_F, \tag{2}$$

where $h_F = |\operatorname{Pic}(\mathcal{O}_F)|$ is the class number, $\omega_F = |(\mathcal{O}_F)^\times_{tors}|$ is the number of roots of unity in F, and $R_F \in \mathbb{R}$ is the regulator. See, e.g., [7, Chapter 5, §1] or [34, §VII.5].

The question naturally arises whether there are formulas similar to (2) for $s = n \in \mathbb{Z}$ other than $s = 0$ (or $s = 1$, which is related to $s = 0$ via the functional equation). To do this, one must find a suitable generalization for the numbers h_F, ω_F, R_F. Many special value conjectures of varying generality go back to this question.

Lichtenbaum proposed formulas in terms of algebraic K-theory in his pioneering work [27]. Later these were also reformulated in terms of p-adic cohomology $H^i(\operatorname{Spec} \mathcal{O}_F[1/p]_{\acute{e}t}, \mathbb{Z}_p(n))$ for $i = 1, 2$ and all primes p; the corresponding formula is known as the **cohomological Lichtenbaum conjecture**; see, for example, [17, §1.7] for the statement and a proof for abelian number fields $F/\mathbb{Q}$. We will not go into details here, since it is more convenient for us to use motivic cohomology instead of working with p-adic cohomology for varying p.

A suitable generalization of R_F are the **higher regulators** considered since the work of Borel [6] and later by Beilinson [1].

We do not attempt to give an adequate historical survey of the subject or to write down all the conjectured formulas; the interested reader may consult, e.g., [25, 16, 21].

Later, Lichtenbaum proposed a new research program known as **Weil-étale cohomology**; see [28, 29, 30, 31]. It suggests that for an arithmetic scheme X the special value of $\zeta(X, s)$ at $s = n \in \mathbb{Z}$ can be expressed in terms of the Weil-étale cohomology, which is a suitable modification of the étale motivic cohomology of X. Flach and Morin in [8] gave a construction of Weil-étale cohomology groups $H^i_{W,c}(X, \mathbb{Z}(n))$ for a proper and regular arithmetic scheme X, and stated a precise conjectural relation of $H^i_{W,c}(X, \mathbb{Z}(n))$ to the special value $\zeta^*(X, n)$.

In [8, §5.8.3] they write down an explicit formula for the case of $X = \operatorname{Spec} \mathcal{O}_F$. For $n \le 0$ and in terms of cohomology groups $H^i(X_{ét}, \mathbb{Z}^c(n))$, it reads

$$\zeta_F^*(n) = \pm \frac{|H^0(X_{ét}, \mathbb{Z}^c(n))|}{|H^{-1}(X_{ét}, \mathbb{Z}^c(n))_{tors}|} \, R_{F,n} \quad \text{for } n \le 0. \tag{3}$$

The definition of $H^i(X_{ét}, \mathbb{Z}^c(n))$ is reviewed below. The regulator $R_{F,n} = R_{\operatorname{Spec} \mathcal{O}_F, n}$ is defined in §6.

By [8, Proposition 5.35], formula (3) holds unconditionally for abelian number fields $F/\mathbb{Q}$, via a reduction to the **Tamagawa number conjecture** of Bloch–Kato–Fontaine–Perrin-Riou.

In particular, if we take $n = 0$, then $\mathbb{Z}^c(0) \cong \mathbb{G}_m[1]$, and $R_{F,0}$ is the usual Dirichlet regulator, so (3) becomes the classical formula (2):

$$\zeta_F^*(0) = \pm \frac{|H^1(\operatorname{Spec} \mathcal{O}_{F,ét}, \mathbb{G}_m)|}{|H^0(\operatorname{Spec} \mathcal{O}_{F,ét}, \mathbb{G}_m)_{tors}|} \, R_F = \pm \frac{|\operatorname{Pic}(\mathcal{O}_F)|}{|(\mathcal{O}_F)^\times_{tors}|} \, R_F,$$

We also mention that Flach and Morin have a similar special value formula for $n > 0$, which includes a correction factor $C(X, n) \in \mathbb{Q}$. In this text we will say nothing about the case of $n > 0$; the reader can consult [8] for more details, and also the subsequent papers [10, 9, 33] which shed light on the nature of the correction factor $C(X, n)$.

For $n < 0$, the author in [2] and [3] extended the work of Flach and Morin [8] to an arbitrary arithmetic scheme X (thus removing the assumption that X is proper or regular). In this text, we would like to work out explicitly the corresponding special value formula for one-dimensional arithmetic schemes.

To state the main result, it is useful to introduce the following terminology.

Definition 1.1. We say that a one-dimensional arithmetic scheme X is **abelian** if each generic point $\eta \in X$ with $\operatorname{char} \kappa(\eta) = 0$ corresponds to an abelian extension $\kappa(\eta)/\mathbb{Q}$.

If X lives in positive characteristic, then it is trivially abelian. The term “abelian” is ad hoc and was suggested by analogy with the notion of **abelian number fields**. Hopefully there is no confusion with the “abelian schemes” that are generalizations of abelian varieties.

Our goal is to prove the following result.

THEOREM 1.2. *For an abelian one-dimensional arithmetic scheme X, the special value of $\zeta(X,s)$ at $s = n < 0$ is given by*

$$\zeta^*(X,n) = \pm 2^\delta \, \frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}| \cdot |H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))|} \, R_{X,n}. \tag{4}$$

Here

- $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ *the étale motivic cohomology from [13];*
- *the correction factor 2^δ is given by*

$$\delta = \delta_{X,n} = \begin{cases} r_1, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \tag{5}$$

  where $r_1 = |X(\mathbb{R})|$ is the number of real places of X,
- $R_{X,n}$ *is a positive real number defined via a regulator map in §6.*

We further conjecture that formula (4) holds for all one-dimensional arithmetic schemes, not necessarily abelian. This is equivalent to the Tamagawa number conjecture for non-abelian number fields (see Remark 7.4).

We give two proofs of (4): first a direct argument in §7 and then an argument in terms of Weil-étale cohomology in §9. In fact, we note that the special value formula is the same as the conjecture $\mathbf{C}(X,n)$ formulated in [3], which is specialized to one-dimensional X and spelled out explicitly.

The purpose of this text is twofold. First, we establish a new special value formula, which generalizes several formulas found in the literature. Second, we review the construction of Weil-étale cohomology $H^i_{W,c}(X, \mathbb{Z}(n))$ from [2] and the special value conjecture from [3] and explain it in the case of one-dimensional schemes. It is not very surprising that a special value formula like (4) exists, but the right cohomological invariants to state it have been suggested by the Weil-étale framework.

This text was inspired in part by the work of Jordan and Poonen [19], which deals with a formula for $\zeta^*(X,1)$, where X is an affine reduced one-dimensional arithmetic scheme. The affine and reduced constraint does not appear in our case because work with different invariants. Since $\zeta(X,s) = \zeta(X_{red}, s)$, the "right" invariants should not distinguish between X and X_{red}, and motivic cohomology satisfies this property.

Notation and conventions

Abelian groups. For an abelian group A, we denote

$$A^D := \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}),$$
$$A^* := \mathrm{Hom}(A, \mathbb{Z}).$$

There is an exact sequence

$$0 \to A^* \to \operatorname{Hom}(A, \mathbb{Q}) \to A^D \to (A_{tors})^D \to 0 \tag{6}$$

Note that for a finite rank group A, the $\mathbb{Z}$-dual A^* is free and has the same rank. If A is finite, then there is a (non-canonical) isomorphism with the $\mathbb{Q}/\mathbb{Z}$-dual $A \cong A^D$, and in particular $|A^D| = |A|$.

Schemes. In this text, X always denotes a **one-dimensional arithmetic scheme**, i.e., a separated scheme of finite type $X \to \operatorname{Spec} \mathbb{Z}$ of Krull dimension 1.

We remark that the restriction that X is abelian (Definition 1.1) is needed only for the proofs of Theorem 1.2 in §7 and §9. Our calculations in §§3, 4, 5, 6, 8 work for any one-dimensional arithmetic scheme X.

Weights. In this text, n always stands for a fixed, *strictly negative* integer.

Motivic cohomology. We will work with a version of étale motivic cohomology defined in terms of **Bloch's cycle complexes**. These were introduced by Bloch in [4] for varieties over fields, and for the version over $\operatorname{Spec} \mathbb{Z}$ see [11, 12].

In short, we let $\Delta^i = \operatorname{Spec} \mathbb{Z}[t_0, \ldots, t_i]/(1 - \sum_i t_i)$ be the algebraic simplex. Denote by $z_n(X, i)$ the group freely generated by algebraic cycles $Z \subset X \times \Delta^i$ of dimension $n + i$ that intersect the faces properly. For $n < 0$ we consider the complex of sheaves on $X_{ét}$

$$\mathbb{Z}^c(n) := z_n(\text{\_}, -\bullet)[2n].$$

The corresponding (hyper)cohomology

$$H^i(X_{ét}, \mathbb{Z}^c(n)) := H^i(R\Gamma(X_{ét}, \mathbb{Z}^c(n)))$$

is what we will call in this text **(étale) motivic cohomology**. For a proper regular arithmetic scheme X of pure dimension d we have

$$\mathbb{Z}^c(n) \cong \mathbb{Z}(d - n)[2d], \tag{7}$$

where $\mathbb{Z}(m)$ is the other motivic complex that usually appears in the literature; see [11, 12] for the definition. To avoid any confusion, all our calculations will be in terms of $\mathbb{Z}^c(n)$.

By [13, Corollary 7.2], the groups $H^i(X_{ét}, \mathbb{Z}^c(n))$ satisfy the **localization property**: if $Z \subset X$ is a closed subscheme and $U = X \setminus Z$ is its closed complement, then there is a distinguished triangle

$$R\Gamma(Z_{ét}, \mathbb{Z}^c(n)) \to R\Gamma(X_{ét}, \mathbb{Z}^c(n)) \to R\Gamma(U_{ét}, \mathbb{Z}^c(n)) \to R\Gamma(Z_{ét}, \mathbb{Z}^c(n))[1],$$

giving a long exact sequence

$$\cdots \to H^i(Z_{ét}, \mathbb{Z}^c(n)) \to H^i(X_{ét}, \mathbb{Z}^c(n)) \to H^i(U_{ét}, \mathbb{Z}^c(n)) \to H^{i+1}(Z_{ét}, \mathbb{Z}^c(n)) \to \cdots \tag{8}$$

This means that $H^i(-, \mathbb{Z}^c(n))$ behaves like (motivic) Borel–Moore homology.

At the level of zeta functions, the localization property corresponds to the identity

$$\zeta(X, s) = \zeta(Z, s)\, \zeta(U, s).$$

For more results on $\mathbb{Z}^c(n)$, we refer the reader to [13].

In general, the groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are very hard to compute. However, they are quite well understood for one-dimensional arithmetic schemes X; see §5 below.

Outline of the paper

In §2 we prove a dévissage lemma that shows how a property that holds for curves over finite fields and for number rings can be generalized to any one-dimensional arithmetic scheme. It is an elementary argument, isolated to avoid repeating the same reasoning in several proofs.

In §3 we calculate the vanishing order of $\zeta(X, s)$ at $s = n < 0$. Then in §4 we calculate the $G_\mathbb{R}$-equivariant cohomology groups of the finite discrete space of complex points $X(\mathbb{C})$. In §5 we put together various well-known results to describe the motivic cohomology groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$. In §6 we define the regulator that appears in the special value formula.

Our first "elementary" proof of the main result is given in §7. Then §8 is devoted to a calculation of the Weil-étale cohomology groups $H^i_{W,c}(X, \mathbb{Z}(n))$ from [2] for one-dimensional X, which we consider an interesting result on its own. We use these calculations in §9 to formulate explicitly the conjecture $\mathbf{C}(X, n)$ from [3], again for one-dimensional X. This is a second, more conceptual proof of the main result, and it explains how we arrived at (4) in the first place.

Finally, we conclude in §10 with a couple of examples showing how our special value formula works.

Acknowledgments

I am grateful to Baptiste Morin for various discussions that led to this text.

2 Dévissage lemma for one-dimensional schemes

The main idea of this paper is to consider a property that holds for spectra of number rings $X = \operatorname{Spec} \mathcal{O}_F$ and curves over finite fields $X/\mathbb{F}_q$, and then generalize it formally to any one-dimensional arithmetic scheme. To this end, in this section we isolate a dévissage argument which will be used repeatedly in the rest of the paper.

LEMMA 2.1. *Let $\mathcal{P}$ be a property of arithmetic schemes of Krull dimension ≤ 1. Suppose that it satisfies the following compatibilities.*

a) $\mathcal{P}(X)$ holds if and only if $\mathcal{P}(X_{red})$ holds.

b) If $X = \coprod_i X_i$ is a finite disjoint union, then $\mathcal{P}(X)$ is equivalent to the conjunction of $\mathcal{P}(X_i)$ for all i.

c) If $U \subset X$ is a dense open subscheme, then $\mathcal{P}(X)$ is equivalent to $\mathcal{P}(U)$.

Suppose that

0) $\mathcal{P}(\operatorname{Spec} \mathbb{F}_q)$ holds for any finite field $\mathbb{F}_q$,

1) $\mathcal{P}(X)$ holds for any smooth curve $X/\mathbb{F}_q$,

2) $\mathcal{P}(\operatorname{Spec} \mathcal{O}_F)$ holds for any number field $F/\mathbb{Q}$.

Then $\mathcal{P}(X)$ holds for any one-dimensional arithmetic scheme X.

Proof. First suppose that $\dim X = 0$. Then, thanks to a), we can assume that X is reduced, and then $X = \coprod_i \operatorname{Spec} \mathbb{F}_{q,i}$ is a finite disjoint union of spectra of finite fields such that $\mathcal{P}(X)$ holds thanks to 0) and b).

Now consider the case of $\dim X = 1$. Again, we can assume that X is reduced. We take the normalization $\nu\colon X' \to X$. This is a birational morphism: there are dense open subschemes $U \subset X$ and $U' \subset X'$ such that $\nu|_{U'} : U' \xrightarrow{\cong} U$ is an isomorphism. Thanks to c), we have

$$\mathcal{P}(X) \iff \mathcal{P}(U) \iff \mathcal{P}(U') \iff \mathcal{P}(X').$$

Therefore, we can assume that X is regular. Now $X = \coprod_i X_i$ is a finite disjoint union of normal integral schemes, so thanks to b), we can assume that X is integral. There are two cases.

- If $X \to \operatorname{Spec} \mathbb{Z}$ lives over a closed point, then it is a smooth curve over $\mathbb{F}_q$, and the claim holds thanks to 1).
- If $X \to \operatorname{Spec} \mathbb{Z}$ is a dominant morphism, consider an open affine neighborhood of the generic point $U \subset X$. Again, $\mathcal{P}(X)$ is equivalent to $\mathcal{P}(U)$, so it suffices to prove the claim for U. We have $U = \operatorname{Spec} \mathcal{O}_{F,S}$ for a number field $F/\mathbb{Q}$ and a finite set of places S, so everything reduces to $\mathcal{P}(\operatorname{Spec} \mathcal{O}_F)$. □

3 Vanishing order of $\zeta(X,s)$ at $s = n < 0$

Definition 3.1 (Numbers r_1 and r_2). Given a one-dimensional arithmetic scheme X, consider the finite discrete space of complex points

$$X(\mathbb{C}) := \operatorname{Hom}(\operatorname{Spec} \mathbb{C}, X).$$

There is a canonical action of the complex conjugation $G_\mathbb{R} := \operatorname{Gal}(\mathbb{C}/\mathbb{R})$ on $X(\mathbb{C})$. The fixed points of this action correspond to the real points $X(\mathbb{R})$, also known as the **real places**. We set $r_1 = |X(\mathbb{R})|$. The non-real places are called **complex places**. They come in conjugate pairs, and we denote their number by $2r_2$.

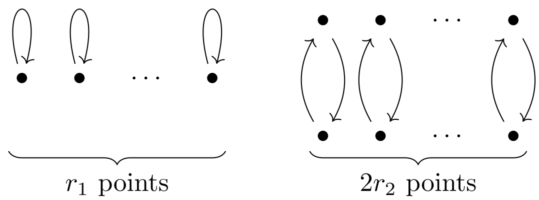

Figure 1: $G_{\mathbb{R}} := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acting on $X(\mathbb{C})$

Equivalently, for a number field $F/\mathbb{Q}$, denote by $r_1(F)$ the number of real embeddings $F \hookrightarrow \mathbb{R}$ and by $r_2(F)$ the number of pairs of complex embeddings $F \hookrightarrow \mathbb{C}$. Then $r_1(F) = r_1$ and $r_2(F) = r_2$ for $X = \operatorname{Spec} \mathcal{O}_F$. In general, for a one-dimensional arithmetic scheme X, we have

$$r_1 = \sum_{\operatorname{char} \kappa(\eta) = 0} r_1(\kappa(\eta)),$$
$$r_2 = \sum_{\operatorname{char} \kappa(\eta) = 0} r_2(\kappa(\eta)),$$

where the sums are over generic points $\eta \in X$ with residue field $\kappa(\eta)$ of characteristic 0.

Proposition 3.2. *Let X be a one-dimensional arithmetic scheme with r_1 real and $2r_2$ complex places. For $n < 0$, the vanishing order of $\zeta(X, s)$ at $s = n$ is given by*

$$d_n = \operatorname{ord}_{s=n} \zeta(X, s) = \begin{cases} r_1 + r_2, & n \text{ even}, \\ r_2, & n \text{ odd}. \end{cases} \tag{9}$$

Proof. For $X = \operatorname{Spec} \mathcal{O}_F$ the claim is a well-known consequence of the functional equation for the Dedekind zeta function [34, §VII.5]. It also holds for $X/\mathbb{F}_q$ since in this case $\zeta(X, s)$ has no zeros or poles at $s = n < 0$ according to [22, pp. 26–27]. We now proceed using Lemma 2.1.

We have $\zeta(X, s) = \zeta(X_{red}, s)$ and $r_{1,2}(X) = r_{1,2}(X_{red})$. If $X = \coprod_i X_i$ is a finite disjoint union, then

$$\operatorname{ord}_{s=n} \zeta(X, s) = \sum_i \operatorname{ord}_{s=n} \zeta(X_i, s),$$
$$r_{1,2}(X) = \sum_i r_{1,2}(X_i),$$

so that the property is compatible with disjoint unions. Finally, if $U \subset X$ is a dense open subscheme, then $Z = X \setminus U$ is a zero-dimensional scheme, and

$$\operatorname{ord}_{s=n} \zeta(X, s) = \operatorname{ord}_{s=n} \zeta(U, s),$$
$$r_{1,2}(X) = r_{1,2}(U),$$

so that the property is compatible with taking dense open subschemes. We conclude that Lemma 2.1 applies. □

4 $G_{\mathbb{R}}$-equivariant cohomology of $X(\mathbb{C})$

Viewing $\mathbb{Z}(n) := (2\pi i)^n\,\mathbb{Z}$ as a constant $G_{\mathbb{R}}$-equivariant sheaf on $X(\mathbb{C})$, we consider the $G_{\mathbb{R}}$-equivariant cohomology groups (resp. Tate cohomology)

$$H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) := H^i\Big(R\Gamma(G_{\mathbb{R}}, R\Gamma_c(X(\mathbb{C}), \mathbb{Z}(n)))\Big),$$
$$\widehat{H}^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) := H^i\Big(R\widehat{\Gamma}(G_{\mathbb{R}}, R\Gamma_c(X(\mathbb{C}), \mathbb{Z}(n)))\Big).$$

Of course, $X(\mathbb{C})$ is just a finite discrete space, so it is not necessary to use cohomology with compact support, but we use this notation for consistency with the general case considered in [2]. Since $\dim X(\mathbb{C}) = 0$, we have

$$H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \cong H^i(G_{\mathbb{R}}, H^0_c(X(\mathbb{C}), \mathbb{Z}(n))),$$
$$\widehat{H}^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \cong \widehat{H}^i(G_{\mathbb{R}}, H^0_c(X(\mathbb{C}), \mathbb{Z}(n))).$$

Proposition 4.1. *Let X be a one-dimensional arithmetic scheme with r_1 real places. Then the $G_{\mathbb{R}}$-equivariant cohomology of $X(\mathbb{C})$ is*

$$\widehat{H}^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}, & i \equiv n\ (2), \\ 0, & i \not\equiv n\ (2); \end{cases} \tag{10}$$

$$H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \cong \begin{cases} 0, & i < 0, \\ \mathbb{Z}^{\oplus d_n}, & i = 0, \\ \widehat{H}^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)), & i \ge 1. \end{cases} \tag{11}$$

Here d_n is the vanishing order given by (9).

Proof. We have

$$H^0_c(X(\mathbb{C}), \mathbb{Z}(n)) \cong \mathbb{Z}(n)^{\oplus r_1} \oplus (\mathbb{Z}(n) \oplus \mathbb{Z}(n))^{\oplus r_2},$$

and the $G_{\mathbb{R}}$-action on the two summands is given by $x \mapsto \overline{x}$ and $(x, y) \mapsto (\overline{y}, \overline{x})$, respectively. (See Figure (1).)

We recall that the Tate cohomology of a finite cyclic group is 2-periodic:

$$\widehat{H}^i(G, A) \cong \begin{cases} \widehat{H}^0(G, A), & i \text{ even}, \\ \widehat{H}_0(G, A), & i \text{ odd}, \end{cases}$$

and the groups $\widehat{H}^0(G, A)$ and $\widehat{H}_0(G, A)$ are given by the exact sequence

$$0 \to \widehat{H}_0(G, A) \to A_G \xrightarrow{N} A^G \to \widehat{H}^0(G, A) \to 0$$

where N is the norm map induced by the action of $\sum_{g \in G} g$.

Therefore, we can consider two cases.

1) $G_{\mathbb{R}}$**-module** $A = \mathbb{Z}(n)$ **with the action via** $x \mapsto \overline{x}$**.**

In this case, we see that

$$A^{G_{\mathbb{R}}} \cong \begin{cases} \mathbb{Z}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

Similarly, it is straightforward to calculate the coinvariants $A_{G_{\mathbb{R}}}$, and

$$\widehat{H}^0(G_{\mathbb{R}}, A) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z}, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \quad \widehat{H}_0(G_{\mathbb{R}}, A) \cong \begin{cases} 0, & n \text{ even}, \\ \mathbb{Z}/2\mathbb{Z}, & n \text{ odd}. \end{cases}$$

2) $G_{\mathbb{R}}$**-module** $A = \mathbb{Z}(n) \oplus \mathbb{Z}(n)$ **with the action via** $(x, y) \mapsto (\overline{y}, \overline{x})$.

In this case $A^{G_{\mathbb{R}}} \cong \mathbb{Z}$ and $\widehat{H}^0(G_{\mathbb{R}}, A) = \widehat{H}_0(G_{\mathbb{R}}, A) = 0$.

Combining these two calculations, we obtain Tate cohomology groups (10). For the usual cohomology (11), we have

$$\begin{aligned} & H^0_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \cong H^0_c(X(\mathbb{C}), \mathbb{Z}(n))^{G_{\mathbb{R}}}, \\ & H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \cong \widehat{H}^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \quad \text{for } i \geq 1. \end{aligned}$$

□

5 Étale motivic cohomology of one-dimensional schemes

In this section we review the structure of the étale motivic cohomology $H^i(X_{ét}, \mathbb{Z}^c(n))$ for one-dimensional X and $n < 0$. What follows is fairly well-known, so we claim no originality here, but we compile the references and state the result for a general one-dimensional arithmetic scheme.

Proposition 5.1. *If X is a one-dimensional arithmetic scheme and $n < 0$, then*

$$H^i(X_{ét}, \mathbb{Z}^c(n)) \cong \begin{cases} 0, & i < -1, \\ \textit{finitely generated of rk } d_n, & i = -1, \\ \textit{finite}, & i = 0, 1, \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}, & i \geq 2, \; i \not\equiv n \; (2), \\ 0, & i \geq 2, \; i \equiv n \; (2). \end{cases} \tag{12}$$

Here d_n is given by (9) and $r_1 = |X(\mathbb{R})|$ is the number of real places of X. Further, if $X = \operatorname{Spec} \mathcal{O}_F$ for a number field $F/\mathbb{Q}$, then

$$H^1(X_{ét}, \mathbb{Z}^c(n)) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}, & n \textit{ even}, \\ 0, & n \textit{ odd}. \end{cases} \tag{13}$$

An important ingredient of our proof is the arithmetic duality [2, Theorem I], which states that if $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finitely generated groups for all $i \in \mathbb{Z}$, then

$$\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \cong H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))^D, \tag{14}$$

where

$$\mathbb{Z}(n) := \mathbb{Q}/\mathbb{Z}(n)[-1] := \bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n}[-1]. \tag{15}$$

Here $\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$ is the modified cohomology with compact support, for which we refer to [15, §2] and [2, Appendix B]. In particular,

$$\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) = H^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \quad \text{if } X(\mathbb{R}) = \varnothing.$$

We recall that $(-)^D$ denotes the group $\operatorname{Hom}(-, \mathbb{Q}/\mathbb{Z})$. We note that (14) is a powerful result, deduced in [2] from the work of Geisser [13].

Proof of Proposition 5.1. We use Lemma 2.1. We will say that $\mathcal{P}(X)$ holds if the motivic cohomology of X has the structure (12).

Let us first consider the case of a finite field $X = \operatorname{Spec} \mathbb{F}_q$. We have

$$H^i(\operatorname{Spec} \mathbb{F}_{q,\acute{e}t}, \mathbb{Z}^c(n)) \cong \begin{cases} \mathbb{Z}/(q^{-n} - 1), & i = 1, \\ 0, & i \neq 1. \end{cases} \tag{16}$$

—see, for example, [14, Example 4.2]. This is related to Quillen's calculation of the K-theory of finite fields [35].

In general, if X is a zero-dimensional arithmetic scheme, then the motivic cohomology of X and X_{red} coincide, so we can assume that X is reduced. Then X is a finite disjoint union of $X_i = \operatorname{Spec} \mathbb{F}_{q_i}$, and

$$H^i(X, \mathbb{Z}^c(n)) = \begin{cases} \text{finite}, & i = 1, \\ 0, & i \neq 1. \end{cases} \tag{17}$$

In particular, $\mathcal{P}(X)$ holds if $\dim X = 0$.

Now we check the compatibility properties for $\mathcal{P}$. If $X = \coprod_i X_i$ is a finite disjoint union, then $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \cong \bigoplus_i H^i(X_{i,\acute{e}t}, \mathbb{Z}^c(n))$, hence the property $\mathcal{P}$ is compatible with disjoint unions.

Similarly, if $U \subset X$ is a dense open subscheme, and $Z = X \setminus U$ its closed complement, then $\dim Z = 0$. We consider the long exact sequence (8). Since the cohomology of Z is concentrated in $i = 1$, we have $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \cong H^i(U_{\acute{e}t}, \mathbb{Z}^c(n))$ for $i \neq 0, 1$, and what is left is an exact sequence

$$0 \to H^0(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to H^0(U_{\acute{e}t}, \mathbb{Z}^c(n)) \to \\ H^1(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \to H^1(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to H^1(U_{\acute{e}t}, \mathbb{Z}^c(n)) \to 0$$

Moreover, $d_n(X) = d_n(U)$. These considerations show that $\mathcal{P}(X)$ and $\mathcal{P}(U)$ are equivalent, and therefore Lemma 2.1 works, and it remains to establish $\mathcal{P}(X)$ for a curve $X/\mathbb{F}_q$ or $X = \operatorname{Spec}\mathcal{O}_F$.

Suppose that $X/\mathbb{F}_q$ is a smooth curve. The groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finitely generated by [14, Proposition 4.3], so that the duality (14) holds. The $\mathbb{Q}/\mathbb{Z}$-dual groups

$$H^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) = \bigoplus_{\ell} H^{i-1}_c(X_{\acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$$

are finite by [20, Theorem 3], and concentrated in $i = 1, 2, 3$ for dimension reasons. It follows that $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ in this case are finite groups concentrated in $i = -1, 0, 1$, and the property $\mathcal{P}(X)$ holds.

It remains to consider the case of $X = \operatorname{Spec}\mathcal{O}_F$. In this case, the finite generation of $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is also known; see, for example, [14, Proposition 4.14]. Therefore, the duality (14) holds. We have $\widehat{H}^i_c(\operatorname{Spec}\mathcal{O}_F[1/p], \mu_{p^r}^{\otimes n}) = 0$ for $i \geq 3$ by Artin–Verdier duality [32, Chapter II, Corollary 3.3], or by [37, p. 268]. Therefore, it follows that $\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) = 0$ for $i \geq 4$, and hence by duality (14), $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) = 0$ for $i \leq -2$.

Now we identify the finite 2-torsion in $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ for $i \geq 2$. By [8, Lemma 6.14], there is an exact sequence

$$\cdots \to H^{i-1}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to \widehat{H}^{i-1}(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \to \widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to H^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to \cdots \quad (18)$$

For $i \leq 0$ we have $H^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) = 0$, and therefore

$$\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \cong \widehat{H}^{i-1}_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}, & i \not\equiv n \ (2), \\ 0, & i \equiv n \ (2). \end{cases}$$

By duality, for $i \geq 2$ we have

$$H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}, & i \not\equiv n \ (2), \\ 0, & i \equiv n \ (2). \end{cases}$$

Now we determine the ranks of $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ for $i = -1, 0, 1$. By [26, Proposition 2.1] the Chern character for $i = -1, 0$

$$K_{-2n-i}(X) \to H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$$

has a finite 2-torsion kernel and cokernel. (Originally, the target group is defined over X_{Zar}, and we identify it with the cohomology on $X_{\acute{e}t}$ using the Beilinson–Lichtenbaum conjecture, which is now a theorem [11, Theorem 1.2]. We further use the isomorphism (7) to identify our motivic cohomology with the one used in [26].)

For $i = -1, 0$ we have therefore

$$\mathrm{rk}_{\mathbb{Z}}\, H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) = \mathrm{rk}_{\mathbb{Z}}\, K_{-2n-i}(X).$$

Together with Borel's calculation of the ranks of $K_m(\mathcal{O}_F)$ in [5], this implies that $H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is a finite group, while

$$\mathrm{rk}_{\mathbb{Z}}\, H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) = d_n = \begin{cases} r_1 + r_2, & n \text{ even}, \\ r_2, & n \text{ odd}. \end{cases}$$

Finally, by [26, p. 179] and (7), we have

$$H^1(X_{\acute{e}t}, \mathbb{Z}^c(n)) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

This concludes the proof. □

6 Regulator for one-dimensional X

Now we explain what is meant by the regulator in our situation.

Definition 6.1. We let the **regulator morphism** be the composition

$$\varrho_{X,n}\colon H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) \xrightarrow{x \mapsto x \otimes 1} H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) \otimes \mathbb{R} \xrightarrow{Reg_{X,n}} H^0_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)),$$

where the map $Reg_{X,n}$ is defined in [3, §2].

The target is the Borel–Moore cohomology defined by

$$H^0_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) := \mathrm{Hom}(H^0_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)), \mathbb{R}).$$

In general, the regulator takes values in Deligne–Beilinson cohomology, but the target simplifies in the case of $n < 0$, as explained in [3, §2].

Remark 6.2. The only relevant group for the regulator is $H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))$, since the cohomology in other degrees is finite by Proposition 5.1.

The general definition in [3, §2] is based on the construction of Kerr, Lewis and Müller-Stach [23] which works at the level of complexes. This is not very important in the one-dimensional case, where the interesting cohomology is concentrated in $i = -1$. The reader can use any other equivalent construction of the regulator for motivic cohomology.

Remark 6.3. If $X = \operatorname{Spec} \mathcal{O}_F$, then $\varrho_{X,n}$ can be identified with the Beilinson regulator map that appears in the special value conjecture of Flach and Morin in [8, §5.8.3].

Lemma 6.4. *For any one-dimensional arithmetic scheme X and $n < 0$, the $\mathbb{R}$-dual to the regulator*

$$Reg^\vee_{X,n} \colon H^0_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n)) \to \operatorname{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})$$

is an isomorphism.

Proof. If $X/\mathbb{F}_q$, then the claim is trivial. For $X = \operatorname{Spec} \mathcal{O}_F$, this is a well-known property of the Beilinson regulator. To apply Lemma 2.1, we need to check compatibility with disjoint unions and passing to a dense open subscheme $U \subset X$. For disjoint unions, this is clear. For a dense open subscheme $U \subset X$, the closed complement $Z = X \setminus U$ has dimension 0, and the localization exact sequence (8) with the long exact sequence for cohomology with compact support yields integral isomorphisms

$$H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) \xrightarrow{\cong} H^{-1}(U_{\acute{e}t}, \mathbb{Z}^c(n)),$$
$$H^0_c(G_\mathbb{R}, U(\mathbb{C}), \mathbb{Z}(n)) \xrightarrow{\cong} H^0_c(G_\mathbb{R}, Z(\mathbb{C}), \mathbb{Z}(n)).$$

We now have a commutative diagram

$$\begin{array}{ccc} H^0_c(G_\mathbb{R}, U(\mathbb{C}), \mathbb{R}(n)) & \xrightarrow{Reg^\vee_{U,n}} & \operatorname{Hom}(H^{-1}(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \\ \downarrow \cong & & \downarrow \cong \\ H^0_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n)) & \xrightarrow{Reg^\vee_{X,n}} & \operatorname{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \end{array}$$

The upper arrow is an isomorphism if and only if the lower arrow is. ☐

Definition 6.5. For a one-dimensional arithmetic scheme X, we define the **regulator** to be

$$R_{X,n} := \operatorname{vol}\Big(\operatorname{coker}\big(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) \xrightarrow{\varrho_{X,n}} H^0_{BM}(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n))\big)\Big),$$

where the volume is taken with respect to the canonical integral structure.

If $X(\mathbb{C}) = \varnothing$, or n is odd and $r_2 = 0$, then $H^0_{BM}(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n)) = 0$, and we set $R_{X,n} = 1$.

Lemma 6.6. *Let X be a one-dimensional arithmetic scheme and $n < 0$. For any dense open subscheme $U \subset X$, we have $R_{X,n} = R_{U,n}$.*

Proof. Follows from the proof of Lemma 6.4. ☐

Proposition 6.7. *Given a one-dimensional arithmetic scheme X and $n < 0$, consider the two-term acyclic complex of real vector spaces*

$$C^\bullet \colon 0 \to \underbrace{H^0_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n))}_{\deg 0} \xrightarrow{Reg^\vee_{X,n}} \underbrace{\operatorname{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})}_{\deg 1} \to 0$$

Then taking the determinant $\det_{\mathbb{R}}(C^\bullet)$ *in the sense of Knudsen and Mumford [24], the image of the canonical map*

$$\det_{\mathbb{Z}} H^0_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} \mathrm{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z})^{-1} \to \\ \det_{\mathbb{R}} H^0_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) \otimes_{\mathbb{R}} \det_{\mathbb{R}} \mathrm{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})^{-1} \xrightarrow{\cong} \mathbb{R}$$

corresponds to $R_{X,n}\,\mathbb{Z} \subset \mathbb{R}$.

Proof. In general, if F and F' are free groups of finite rank d, and

$$C^\bullet \colon 0 \to F \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\phi} F' \otimes_{\mathbb{Z}} \mathbb{R} \to 0$$

is a two-term acyclic complex of real vector spaces, then the image of

$$\mathbb{Z} \cong \det_{\mathbb{Z}} F \otimes_{\mathbb{Z}} (\det_{\mathbb{Z}} F')^{-1} \to \det_{\mathbb{R}}(F \otimes_{\mathbb{Z}} \mathbb{R}) \otimes_{\mathbb{R}} \det_{\mathbb{R}}(F' \otimes_{\mathbb{Z}} \mathbb{R})^{-1} = \det_{\mathbb{R}}(C^\bullet) \xrightarrow{\cong} \mathbb{R}$$

corresponds to $D\mathbb{Z} \subset \mathbb{R}$, where D is the determinant of ϕ with respect to the bases induced by $\mathbb{Z}$-bases of F and F'. This follows from the explicit description of the isomorphism $\det_{\mathbb{R}}(C^\bullet) \xrightarrow{\cong} \mathbb{R}$ from [24, p. 33]: it is

$$\det_{\mathbb{R}}(F \otimes_{\mathbb{Z}} \mathbb{R}) \otimes_{\mathbb{R}} \det_{\mathbb{R}}(F' \otimes_{\mathbb{Z}} \mathbb{R})^{-1} \xrightarrow{\det_{\mathbb{R}}(\phi)} \det_{\mathbb{R}}(F' \otimes_{\mathbb{Z}} \mathbb{R}) \otimes_{\mathbb{R}} \det_{\mathbb{R}}(F' \otimes_{\mathbb{Z}} \mathbb{R})^{-1} \xrightarrow{\cong} \mathbb{R}$$

where the last arrow is the canonical pairing.

Therefore, in our situation, the image of

$$\det_{\mathbb{Z}} H^0_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} \mathrm{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z})^{-1}$$

is $D\mathbb{Z} \subset \mathbb{R}$, where D is the determinant of $Reg^\vee_{X,n}$ considered with respect to the bases induced by $\mathbb{Z}$-bases of $H^0_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))$ and $\mathrm{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z})$. Dually, $D = R_{X,n}$. □

7 Direct proof of the special value formula

In this section we explain how to prove our special value formula directly by combining the known special value formulas for $X = \operatorname{Spec} \mathcal{O}_F$ and curves over finite fields $X/\mathbb{F}_q$ via localization.

LEMMA 7.1. *Let* $n < 0$.

0) If X *is a zero-dimensional arithmetic scheme, then*

$$\zeta(X, n) = \pm \frac{1}{|H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))|}.$$

1) If $X/\mathbb{F}_q$ *is a curve over a finite field, then*

$$\zeta(X, n) = \pm \frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))| \cdot |H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))|}.$$

2) If $X = \operatorname{Spec} \mathcal{O}_F$ for an abelian number field $F/\mathbb{Q}$, then

$$\zeta(X,n) = \pm \frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))|} R_{X,n}.$$

In particular, formula (4) holds in these cases.

Proof. In part 0), motivic cohomology and the zeta function do not distinguish between X and X_{red}, so we can assume that X is a finite disjoint union of $\operatorname{Spec} \mathbb{F}_{q_i}$. Thanks to (16),

$$\zeta(X,n) = \prod_i \frac{1}{1-q_i^{-n}} = \pm \prod_i \frac{1}{|H^1(X_{i,\acute{e}t}, \mathbb{Z}^c(n))|} = \pm \frac{1}{|H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))|}.$$

Note that this is formula (4) since $\delta = 0$ in this case and $H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) = H^0(X_{\acute{e}t}, \mathbb{Z}^c(n)) = 0$ by (16).

For part 1), we refer the reader to [3, §5]. Part 2) follows from [8, Proposition 5.35]. The formula is equivalent to (4), since $2^\delta = |H^1(X_{\acute{e}t}, \mathbb{Z}(n))|$ by (13). □

Remark 7.2. The special value at $s = 0$ is not necessarily a rational number:

$$\zeta^*(\operatorname{Spec} \mathbb{F}_q, 0) = \lim_{s\to 0} \frac{s}{1-q^{-s}} = \frac{1}{\log q}.$$

Moreover,

$$H^i(\operatorname{Spec} \mathbb{F}_{q,\acute{e}t}, \mathbb{Z}^c(0)) = \begin{cases} \mathbb{Z}, & i = 1, \\ \mathbb{Q}/\mathbb{Z}, & i = 2, \\ 0, & i \neq 1,2. \end{cases}$$

This toy example already shows that it is important that we focus on the case of $n < 0$.

Lemma 7.3. *Let X be a one-dimensional arithmetic scheme and let $U \subset X$ be a dense open subscheme. Then the special value formula (4) for X is equivalent to the corresponding formula for U.*

Proof. Let $Z = X \setminus U$ be the zero-dimensional complement. We have

$$\zeta(X,n) = \zeta(U,n)\,\zeta(Z,n),$$

where

$$\zeta(X,n) \stackrel{?}{=} \pm 2^\delta \frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(n)|}{|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}| \cdot |H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))|} R, \tag{19}$$

$$\zeta(U,n) \stackrel{?}{=} \pm 2^\delta \frac{|H^0(U_{\acute{e}t}, \mathbb{Z}^c(n)|}{|H^{-1}(U_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}| \cdot |H^1(U_{\acute{e}t}, \mathbb{Z}^c(n))|} R, \tag{20}$$

$$\zeta(Z,n) = \pm \frac{1}{|H^1(Z_{\acute{e}t}, \mathbb{Z}^c(n))|}.$$

Here $\delta = \delta_{X,n} = \delta_{U,n}$, and $R = R_{X,n} = R_{U,n}$ (see Lemma 6.6). We note that $|H^{-1}(X_{ét}, \mathbb{Z}^c(n))_{tors}| = |H^{-1}(U_{ét}, \mathbb{Z}^c(n))_{tors}|$. On the other hand, the exact sequence of finite groups

$$0 \to H^0(X_{ét}, \mathbb{Z}^c(n)) \to H^0(U_{ét}, \mathbb{Z}^c(n)) \to \\ H^1(Z_{ét}, \mathbb{Z}^c(n)) \to H^1(X_{ét}, \mathbb{Z}^c(n)) \to H^1(U_{ét}, \mathbb{Z}^c(n)) \to 0 \quad (21)$$

gives

$$\frac{|H^0(X_{ét}, \mathbb{Z}^c(n))|}{|H^1(X_{ét}, \mathbb{Z}^c(n))|} = \frac{|H^0(U_{ét}, \mathbb{Z}^c(n))|}{|H^1(U_{ét}, \mathbb{Z}^c(n))|} \cdot \frac{1}{|H^1(Z_{ét}, \mathbb{Z}^c(n))|}.$$

From this we see that (19) and (20) are equivalent. □

The above Lemmas 7.1 and 7.3 together with Lemma 2.1 now prove Theorem 1.2 from the introduction.

Remark 7.4. Our proof of Lemma 7.1 uses [8, Proposition 5.35], which in turn reduces to the Tamagawa number conjecture for abelian $F/\mathbb{Q}$. The non-abelian version of Theorem 1.2 is therefore equivalent to the corresponding conjecture for non-abelian $F/\mathbb{Q}$.

Remark 7.5. Note that $\zeta(\operatorname{Spec} \mathbb{F}_q, n) = \frac{1}{1-q^{-n}} < 0$. Thus, if we remove m closed points from X, the sign of $\zeta^*(X, n)$ changes by $(-1)^m$. It is not hard to figure out the sign in any concrete example; however, it is not so clear in what terms to write the general expression for the sign.

8 Weil-étale cohomology of one-dimensional arithmetic schemes

In this section we calculate Weil-étale cohomology groups $H^i_{W,c}(X, \mathbb{Z}(n))$ for $n < 0$, as defined in [2]. Let us briefly recall the construction. In general, let X be an arithmetic scheme with finitely generated motivic cohomology $H^i(X_{ét}, \mathbb{Z}^c(n))$. The construction is carried out in two steps.

- **Step 1**. Consider the morphism in the derived category $\mathbf{D}(\mathbb{Z})$

  $$\alpha_{X,n} \colon R\operatorname{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \to R\Gamma_c(X_{ét}, \mathbb{Z}(n))$$

  determined at the level of cohomology, using the arithmetic duality (14), by

  $$H^i(\alpha_{X,n}) \colon \operatorname{Hom}(H^{2-i}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}) \xrightarrow{\mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}} H^{2-i}(X_{ét}, \mathbb{Z}^c(n))^D \\ \xleftarrow{\cong} \widehat{H}^i_c(X_{ét}, \mathbb{Z}(n)) \to H^i_c(X_{ét}, \mathbb{Z}(n)). \quad (22)$$

The complex $R\Gamma_{fg}(X,\mathbb{Z}(n))$ is defined as a cone of $\alpha_{X,n}$:

$$R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) \\ \to R\Gamma_{fg}(X_{\acute{e}t},\mathbb{Z}(n)) \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1])$$

The groups

$$H^i_{fg}(X,\mathbb{Z}(n)) := H^i(R\Gamma_{fg}(X,\mathbb{Z}(n)))$$

are finitely generated for all $i\in\mathbb{Z}$, vanish for $i\ll 0$, and finite 2-torsion for $i\gg 0$. For the details we refer to [2, §5].

- **Step 2**. We consider a canonical morphism i^*_∞ in the derived category $\mathbf{D}(\mathbb{Z})$ which is torsion and yields a commutative diagram

$$\begin{array}{ccc} R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X,\mathbb{Z}(n)) \\ \downarrow u^*_\infty & \swarrow i^*_\infty & \\ R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) & & \end{array}$$

—see [2, §§6,7] for more details. Weil-étale cohomology with compact support is defined as a mapping fiber of i^*_∞:

$$R\Gamma_{W,c}(X,\mathbb{Z}(n)) \to R\Gamma_{fg}(X,\mathbb{Z}(n)) \xrightarrow{i^*_\infty} R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \to [1]$$

The resulting groups

$$H^i_{W,c}(X,\mathbb{Z}(n)) := H^i(R\Gamma_{W,c}(X,\mathbb{Z}(n)))$$

are finitely generated and vanish for $i\notin[0,2\dim X+1]$. We refer to [2, §7] for the general properties.

Here we calculate $H^i_{W,c}(X,\mathbb{Z}(n))$ for one-dimensional X.

Proposition 8.1. *Let X be a one-dimensional arithmetic scheme and $n<0$.*

0) $H^i_{W,c}(X,\mathbb{Z}(n))=0$ for $i\neq 1,2,3$.

1) There is a short exact sequence

$$0\to \underbrace{H^0_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))}_{\cong\mathbb{Z}^{\oplus d_n}} \to H^1_{W,c}(X,\mathbb{Z}(n)) \to T_1 \to 0 \qquad (23)$$

in which T_1 sits in a short exact sequence of finite groups

$$0\to \widehat{H}^0_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \to H^1(X_{\acute{e}t},\mathbb{Z}^c(n))^D \to T_1\to 0$$

In particular, $H^1_{W,c}(X,\mathbb{Z}(n))$ is finitely generated of rank d_n, and

$$|T_1| = \frac{1}{2^\delta}\cdot|H^1(X_{\acute{e}t},\mathbb{Z}^c(n))|,$$

where δ is defined by (5).

2) There is an isomorphism of finitely generated groups

$$H^2_{W,c}(X,\mathbb{Z}(n)) \cong \underbrace{H^{-1}(X_{ét},\mathbb{Z}^c(n))^*}_{\cong \mathbb{Z}^{\oplus d_n}} \oplus \underbrace{H^0(X_{ét},\mathbb{Z}^c(n))^D}_{finite}.$$

3) There is an isomorphism of finite groups

$$H^3_{W,c}(X,\mathbb{Z}(n)) \cong (H^{-1}(X_{ét},\mathbb{Z}^c(n))_{tors})^D.$$

We recall that $A^D := \mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z})$ and $A^* := \mathrm{Hom}(A,\mathbb{Z})$.

Proof. From the definition of $R\Gamma_{fg}(X,\mathbb{Z}(n))$ we have a long exact sequence

$$\cdots \to \mathrm{Hom}(H^{2-i}(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}) \xrightarrow{H^i(\alpha_{X,n})} H^i_c(X_{ét},\mathbb{Z}(n)) \\ \to H^i_{fg}(X,\mathbb{Z}(n)) \to \mathrm{Hom}(H^{1-i}(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}) \to \cdots \quad (24)$$

Our calculations of motivic cohomology in Proposition 5.1 give

$$\mathrm{Hom}(H^i(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}) = 0 \quad \text{for } i \neq -1,$$

and further by the definition of $\mathbb{Z}(n)$ in (15),

$$H^i_c(X_{ét},\mathbb{Z}(n)) = 0 \quad \text{for } i \leq 0.$$

This implies that $H^i_{fg}(X,\mathbb{Z}(n)) = 0$ for $i \leq 0$. Since $H^i_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) = 0$ for $i < 0$, we see from the exact sequence

$$\cdots \to H^i_{W,c}(X,\mathbb{Z}(n)) \to H^i_{fg}(X,\mathbb{Z}(n)) \xrightarrow{H^i(i^*_\infty)} H^i_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \\ \to H^{i+1}_{W,c}(X,\mathbb{Z}(n)) \to \cdots \quad (25)$$

that $H^i_{W,c}(X,\mathbb{Z}(n)) = 0$ for $i \leq 0$.

For $i = 1$, the exact sequence (24) shows that $H^1_c(X_{ét},\mathbb{Z}(n)) \to H^1_{fg}(X_{ét},\mathbb{Z}(n))$ is an isomorphism. Consequently, we see that $\ker H^1(i^*_\infty) \cong \ker H^1(u^*_\infty)$:

$$\begin{array}{ccc} H^1_c(X_{ét},\mathbb{Z}(n)) & \xrightarrow{\cong} & H^1_{fg}(X,\mathbb{Z}(n)) \\ {\scriptstyle H^1(u^*_\infty)}\downarrow & \swarrow {\scriptstyle H^1(i^*_\infty)} & \\ H^1_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) & & \end{array}$$

From long exact sequences (25) and (18), we obtain short exact sequences

$$0 \to H^0_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \to H^1_{W,c}(X,\mathbb{Z}(n)) \longrightarrow \ker H^1(i^*_\infty) \to 0$$

$$0 \to \widehat{H}^0_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \longrightarrow \widehat{H}^1_c(X_{ét},\mathbb{Z}(n)) \longrightarrow \ker H^1(u^*_\infty) \to 0$$

Since $\ker H^1(i_\infty^*) \cong \ker H^1(u_\infty^*)$, this is part 1) of the proposition.

We proceed to compute $H^i_{W,c}(X, \mathbb{Z}(n))$ for $i \geq 2$. It is more convenient to do this without passing explicitly through $H^i_{fg}(X, \mathbb{Z}(n))$. Consider the morphism of complexes

$$\widehat{\alpha}_{X,n} \colon R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \to R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)),$$

defined in the same way as $\alpha_{X,n}$ in (22), only without the final projection from $\widehat{H}^i_c$ to H^i_c:

$$\begin{aligned} H^i(\widehat{\alpha}_{X,n}) \colon \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) &\xrightarrow{\mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}} H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))^D \\ &\xleftarrow{\cong} \widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)). \end{aligned}$$

The relation between $\widehat{\alpha}_{X,n}$ and $\alpha_{X,n}$ is given by

$$\begin{array}{ccc} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow{\widehat{\alpha}_{X,n}} & R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \\ & \searrow^{\alpha_{X,n}} & \downarrow \\ & & R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \end{array}$$

Here the vertical arrow comes from the definition of modified étale cohomology with compact support and it sits in an exact triangle

$$R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \xrightarrow{\widehat{u}_\infty^*} R\widehat{\Gamma}_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \to \cdots [1]$$

—see [8, Lemma 6.14]. From the definition of $\widehat{\alpha}_{X,n}$ and the exact sequence (6), we calculate

$$\begin{aligned} \ker H^i(\widehat{\alpha}_{X,n}) &= H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))^*, \\ \mathrm{coker}\, H^i(\widehat{\alpha}_{X,n}) &\cong (H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors})^D. \end{aligned}$$

We denote a cone of $\widehat{\alpha}_{X,n}$ by $R\widehat{\Gamma}_{fg}(X, \mathbb{Z}(n))$ and set

$$\widehat{H}^i_{fg}(X, \mathbb{Z}(n)) := H^i(R\widehat{\Gamma}_{fg}(X, \mathbb{Z}(n))),$$

so that there is a long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) &\xrightarrow{H^i(\widehat{\alpha}_{X,n})} \\ \widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to \widehat{H}^i_{fg}(X, \mathbb{Z}(n)) &\to \mathrm{Hom}(H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \to \cdots \end{aligned}$$

The corresponding short exact sequences

$$0 \to \mathrm{coker}\, H^i(\widehat{\alpha}_{X,n}) \to \widehat{H}^i_{fg}(X, \mathbb{Z}(n)) \to \ker H^{i+1}(\widehat{\alpha}_{X,n}) \to 0$$

are split, since $\ker H^{i+1}(\widehat{\alpha}_{X,n})$ is a free group. Therefore, we have

$$\widehat{H}^i_{fg}(X,\mathbb{Z}(n)) \cong H^{1-i}(X_{\acute{e}t},\mathbb{Z}^c(n))^* \oplus (H^{2-i}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors})^D.$$

There is a commutative diagram with distinguished rows and columns

$$\begin{array}{ccccccc}
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) & \xrightarrow{\widehat{\alpha}_{X,n}} & R\widehat{\Gamma}_c(X_{\acute{e}t},\mathbb{Z}(n)) & \longrightarrow & R\widehat{\Gamma}_{fg}(X,\mathbb{Z}(n)) & \longrightarrow & [+1] \\
\downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) & \xrightarrow{\alpha_{X,n}} & R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X,\mathbb{Z}(n)) & \longrightarrow & [+1] \\
\downarrow & & \downarrow{\scriptstyle \widehat{u}^*_\infty} & & \downarrow{\scriptstyle \widehat{i}^*_\infty} & & \downarrow \\
0 & \longrightarrow & R\widehat{\Gamma}_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) & \xrightarrow{id} & R\widehat{\Gamma}_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]) & \xrightarrow{\widehat{\alpha}_{X,n}[1]} & R\widehat{\Gamma}_c(X_{\acute{e}t},\mathbb{Z}(n))[1] & \longrightarrow & R\widehat{\Gamma}_{fg}(X,\mathbb{Z}(n))[1] & \longrightarrow & [+2]
\end{array}$$

Here $\widehat{u}^*_\infty$ (resp. $\widehat{i}^*_\infty$) is defined as the composition of the canonical morphism u^*_∞ (resp. i^*_∞) with the projection to the Tate cohomology

$$\pi\colon R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) \to R\widehat{\Gamma}_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)).$$

In our case of one-dimensional X, we know that $H^i(\pi)$ is an isomorphism for $i \geq 1$ (cf. [2, Proposition 3.2]). Therefore, the five-lemma applied to

$$\begin{array}{ccccccc}
R\Gamma_{W,c}(X,\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X,\mathbb{Z}(n)) & \xrightarrow{i^*_\infty} & R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) & \longrightarrow & \cdots[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle f[1]} \\
R\widehat{\Gamma}_{fg}(X,\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X,\mathbb{Z}(n)) & \xrightarrow{\widehat{i}^*_\infty} & R\widehat{\Gamma}_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) & \longrightarrow & \cdots[1]
\end{array}$$

shows that for $i \geq 2$ holds

$$H^i_{W,c}(X,\mathbb{Z}(n)) \cong \widehat{H}^i_{fg}(X,\mathbb{Z}(n)) \cong H^{1-i}(X_{\acute{e}t},\mathbb{Z}^c(n))^* \oplus (H^{2-i}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors})^D.$$

Our calculations of motivic cohomology in Proposition 5.1 yield

$$\begin{aligned}
&H^2_{W,c}(X,\mathbb{Z}(n)) \cong H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))^* \oplus H^0(X_{\acute{e}t},\mathbb{Z}^c(n))^D, \\
&H^3_{W,c}(X,\mathbb{Z}(n)) \cong (H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors})^D, \\
&H^i_{W,c}(X,\mathbb{Z}(n)) = 0 \text{ for } i \geq 4.
\end{aligned}$$

□

Remark 8.2. A priori, the short exact sequence (23) need not split. This will not bother us for the determinant calculations in §9 below.

Remark 8.3. The groups $H^i_{W,c}(X,\mathbb{Z}(n))$ for $X = \operatorname{Spec}\mathcal{O}_F$ are already calculated in [8, §5.8.3]. The result is (using the identification (7))

$$H^i_{W,c}(X,\mathbb{Z}(n)) \cong \begin{cases} \mathbb{Z}^{\oplus d_n}, & i = 1, \\ H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))^* \oplus H^0(X_{\acute{e}t},\mathbb{Z}^c(n))^D, & i = 2, \\ (H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors})^D, & i = 3, \\ 0, & i \neq 1,2,3. \end{cases} \tag{26}$$

Our calculation generalizes this. What may look puzzling is the answer for $H^1_{W,c}(X,\mathbb{Z}(n))$ given by Proposition 8.1. In the case of $X = \operatorname{Spec} \mathcal{O}_F$ we have, according to (13), that $H^1(X_{\acute{e}t}, \mathbb{Z}^c(n)) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}$ for even n, and hence $T_1 = 0$, which agrees with (26).

Intuitively, the arithmetically interesting cohomology $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ for $X = \operatorname{Spec} \mathcal{O}_F$ is concentrated in degrees $i = -1, 0$. The groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ for $i \ge 1$ do not contain any interesting information: they are finite 2-torsion, coming from the real places of F. The transition to Weil-étale cohomology eliminates this 2-torsion. On the other hand, the group $H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))$ for a curve over a finite field $X/\mathbb{F}_q$ is nontrivial and contains arithmetic information. The finite group T_1 appearing in the statement removes the 2-torsion coming from the real places of X.

Remark 8.4. For a curve over a finite field $X/\mathbb{F}_q$, all groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finite, and our calculation gives $H^i_{W,c}(X,\mathbb{Z}(n)) \cong H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))^D$. This is true for any variety over a finite field $X/\mathbb{F}_q$ and $n < 0$, under the assumption of finite generation of $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$; see [2, Proposition 7.7].

Remark 8.5. It is conjectured in [3, §3] that

$$\operatorname{ord}_{s=n} \zeta(X,s) = \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n)).$$

In this case

$$\begin{aligned} &\operatorname{rk}_{\mathbb{Z}} H^1_{W,c}(X,\mathbb{Z}(n)) = \operatorname{rk}_{\mathbb{Z}} H^2_{W,c}(X,\mathbb{Z}(n)) = d_n, \\ &\operatorname{rk}_{\mathbb{Z}} H^3_{W,c}(X,\mathbb{Z}(n)) = 0, \end{aligned}$$

so the conjecture holds by Proposition 3.2.

9 Weil-étale proof of the special value formula

Now we explicitly write down the special value conjecture $\mathbf{C}(X,n)$ from [3, §4]. To do this, consider the canonical isomorphism

$$\begin{aligned} \lambda\colon \mathbb{R} &\xrightarrow{\cong} \bigotimes_{i\in\mathbb{Z}} (\det_{\mathbb{R}} H^i_{W,c}(X,\mathbb{R}(n)))^{(-1)^i} \\ &\xrightarrow{\cong} \Bigl(\bigotimes_{i\in\mathbb{Z}} (\det_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n)))^{(-1)^i}\Bigr) \otimes_{\mathbb{Z}} \mathbb{R} \\ &\xrightarrow{\cong} (\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{R}, \end{aligned}$$

where the first isomorphism $\mathbb{R} \cong \bigotimes_{i\in\mathbb{Z}} (\det_{\mathbb{R}} H^i_{W,c}(X,\mathbb{R}(n)))^{(-1)^i}$ comes from the regulator, as explained below.

In our case, we are interested in the determinant of the Weil-étale complex

$$\begin{aligned}\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) &\cong \bigotimes_{i\in\mathbb{Z}} \det_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n))^{(-1)^i} \\ = \det_{\mathbb{Z}} H^1_{W,c}(X,\mathbb{Z}(n))^{-1} &\otimes_{\mathbb{Z}} \det_{\mathbb{Z}} H^2_{W,c}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} H^3_{W,c}(X,\mathbb{Z}(n))^{-1}.\end{aligned}$$

Using the calculations from Proposition 8.1,

$$\begin{aligned}\det_{\mathbb{Z}} H^1_{W,c}(X,\mathbb{Z}(n)) &\cong \det_{\mathbb{Z}} H^0_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} T_1, \\ \det_{\mathbb{Z}} H^2_{W,c}(X,\mathbb{Z}(n)) &\cong \det_{\mathbb{Z}} H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))^* \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} H^0(X_{\acute{e}t},\mathbb{Z}^c(n))^D, \\ \det_{\mathbb{Z}} H^3_{W,c}(X,\mathbb{Z}(n)) &\cong \det_{\mathbb{Z}}(H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors})^D.\end{aligned}$$

So we have an isomorphism (up to sign ± 1, after rearranging the terms)

$$\begin{aligned}&\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \cong \\ &\qquad = \det_{\mathbb{Z}} H^0_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))^{-1} \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))^* \otimes_{\mathbb{Z}} \\ &\det_{\mathbb{Z}}(T_1)^{-1} \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} H^0(X_{\acute{e}t},\mathbb{Z}^c(n))^D \otimes_{\mathbb{Z}} \det_{\mathbb{Z}}((H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors})^D)^{-1}.\end{aligned}$$

Recall that T_1, $H^0(X_{\acute{e}t},\mathbb{Z}^c(n))^D$, $(H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors})^D$ are finite groups, while the groups $H^0_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))$ and $H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))^*$ are free of rank d_n. Now we consider the canonical trivialization

$$(\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{R} \cong \bigotimes_{i\in\mathbb{Z}} \det_{\mathbb{R}}(H^i_{W,c}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{R}) \cong \mathbb{R}$$

via the regulator morphism

$$\begin{array}{ccc} H^0_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \otimes \mathbb{R} & & \operatorname{Hom}(H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Z}) \otimes \mathbb{R} \\ \downarrow \cong & & \downarrow \cong \\ H^0_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{R}(n)) & \xrightarrow[\cong]{Reg^\vee_{X,n}} & \operatorname{Hom}(H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}) \end{array}$$

PROPOSITION 9.1. *Under the above trivialization,* $\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \subset \mathbb{R}$ *corresponds to* $\alpha^{-1}\mathbb{Z} \subset \mathbb{R}$*, where*

$$\begin{aligned}\alpha &= \frac{|H^0(X_{\acute{e}t},\mathbb{Z}^c(n))^D|}{|T_1| \cdot |(H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors})^D|}\, R_{X,n} \\ &= 2^\delta \, \frac{|H^0(X_{\acute{e}t},\mathbb{Z}^c(n))|}{|H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors}| \cdot |H^1(X_{\acute{e}t},\mathbb{Z}^c(n))|}\, R_{X,n},\end{aligned}$$

the number δ *is given by* (5)*, and* $R_{X,n}$ *is the regulator from Definition 6.5.*

Proof. For the finite groups T_1, $H^0(X_{\acute{e}t},\mathbb{Z}^c(n))^D$, $(H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors})^D$, this is [3, Lemma A.5]. For the free groups $H^0_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))$ and $H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))^*$, on the other hand, this is Proposition 6.7 (now our groups sit in degrees 1 and 2, so the determinant gets inverted). ∎

We recall that Conjecture $\mathbf{C}(X,n)$ from [3, §4] states that the canonical embedding $\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \subset \mathbb{R}$ corresponds to $\zeta^*(X,n)^{-1}\,\mathbb{Z} \subset \mathbb{R}$.

PROPOSITION 9.2. *Let X be a one-dimensional arithmetic scheme and $n < 0$. Then the special value conjecture $\mathbf{C}(X,n)$ stated in [3] is equivalent to formula* (4).

In [3, §7] it is already proved (using essentially the same localization idea as in this text) that $\mathbf{C}(X,n)$ holds unconditionally for an abelian one-dimensional arithmetic scheme X. Together with the proposition above, this proves Theorem 1.2 from the introduction.

10 A couple of examples

We conclude with two examples that illustrate how localization arguments work. The first is rather general and consists in specifying §7 to the case of a non-maximal order in a number field.

EXAMPLE 10.1. Let $\mathcal{O} \subset \mathcal{O}_F$ be a non-maximal order in a number field $F/\mathbb{Q}$. Denote $X = \operatorname{Spec}\mathcal{O}$ and $X' = \operatorname{Spec}\mathcal{O}_F$. Geometrically, $\nu\colon X' \to X$ is the normalization. There exist open dense subschemes $U \subset X$ and $U' \subset X'$ such that ν induces an isomorphism $U' \cong U$. If we denote the corresponding closed complements by $Z = X \setminus U$ and $Z' = X' \setminus U'$, then we have

$$\zeta_{\mathcal{O}}(s) = \frac{\zeta(Z,s)}{\zeta(Z',s)}\,\zeta_F(s).$$

For this identity formulated in classical terms of algebraic number theory, see, for example, [18]. In particular,

$$\zeta_{\mathcal{O}}^*(n) = \pm\frac{|H^1(Z'_{\acute{e}t},\mathbb{Z}^c(n))|}{|H^1(Z_{\acute{e}t},\mathbb{Z}^c(n))|}\,\zeta_F^*(n).$$

Now our special value conjectures for $\zeta_{\mathcal{O}}^*(n)$ and $\zeta_F^*(n)$ take the form

$$\zeta_{\mathcal{O}}^*(n) \stackrel{?}{=} \pm 2^\delta\,\frac{|H^0(X_{\acute{e}t},\mathbb{Z}^c(n))|}{|H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors}|\cdot|H^1(X_{\acute{e}t},\mathbb{Z}^c(n))|}\,R, \tag{27}$$

$$\zeta_F^*(n) \stackrel{?}{=} \pm 2^\delta\,\frac{|H^0(X'_{\acute{e}t},\mathbb{Z}^c(n))|}{|H^{-1}(X'_{\acute{e}t},\mathbb{Z}^c(n))_{tors}|\cdot|H^1(X'_{\acute{e}t},\mathbb{Z}^c(n))|}\,R. \tag{28}$$

Here $|H^{-1}(X_{\acute{e}t},\mathbb{Z}^c(n))_{tors}| = |H^{-1}(X'_{\acute{e}t},\mathbb{Z}^c(n))_{tors}|$, and the exact sequences of finite groups

$$0 \to H^0(X_{\acute{e}t},\mathbb{Z}^c(n)) \to H^0(U_{\acute{e}t},\mathbb{Z}^c(n)) \to H^1(Z_{\acute{e}t},\mathbb{Z}^c(n)) \to H^1(X_{\acute{e}t},\mathbb{Z}^c(n)) \to H^1(U_{\acute{e}t},\mathbb{Z}^c(n)) \to 0$$

$$0 \to H^0(X'_{ét}, \mathbb{Z}^c(n)) \to H^0(U'_{ét}, \mathbb{Z}^c(n)) \to \\ H^1(Z'_{ét}, \mathbb{Z}^c(n)) \to H^1(X'_{ét}, \mathbb{Z}^c(n)) \to H^1(U'_{ét}, \mathbb{Z}^c(n)) \to 0$$

give us

$$\frac{|H^1(Z'_{ét}, \mathbb{Z}^c(n))|}{|H^1(Z_{ét}, \mathbb{Z}^c(n))|} = \frac{|H^1(X'_{ét}, \mathbb{Z}^c(n))|}{|H^1(X_{ét}, \mathbb{Z}^c(n))|} \cdot \frac{|H^0(X_{ét}, \mathbb{Z}^c(n))|}{|H^0(X'_{ét}, \mathbb{Z}^c(n))|},$$

which implies that the formulas (27) and (28) are equivalent.

The second example is suggested by [19, §7].

Example 10.2. Let p be an odd prime. Consider the affine scheme

$$X = \operatorname{Spec}(\mathbb{Z}[1/2] \times_{\mathbb{F}_p} \mathbb{F}_p[t]) = \operatorname{Spec} \mathbb{Z}[1/2] \underset{\operatorname{Spec} \mathbb{F}_p}{\sqcup} \mathbb{A}^1_{\mathbb{F}_p}$$

obtained from $\operatorname{Spec} \mathbb{Z}[1/p]$ and $\mathbb{A}^1_{\mathbb{F}_p} = \operatorname{Spec} \mathbb{F}_p[t]$ by gluing together the points corresponding to the prime ideals $(p) \subset \mathbb{Z}[1/2]$ and $(t) \subset \mathbb{F}_p[t]$:

$$\mathbb{Z}[1/2] \times_{\mathbb{F}_p} \mathbb{F}_p[t] = \{(a, f) \in \mathbb{Z}[1/2] \times \mathbb{F}_p[t] \mid a \equiv f(0) \pmod p\}.$$

If we take odd $n < 0$, then there is no regulator. Let us consider $n = -3$.

First, recall some calculations of the motivic cohomology of $\operatorname{Spec} \mathbb{Z}$. Using [26, Proposition 2.1] and known calculations of the K-groups of $\mathbb{Z}$ (see Weibel's survey [38]), we get

$$\begin{aligned} H^{-1}(\operatorname{Spec} \mathbb{Z}_{ét}, \mathbb{Z}^c(-3)) &\cong K_7(\mathbb{Z}) \cong \mathbb{Z}/240\mathbb{Z}, \\ H^0(\operatorname{Spec} \mathbb{Z}_{ét}, \mathbb{Z}^c(-3)) &\cong \mathbb{Z}/2\mathbb{Z}, \\ H^1(\operatorname{Spec} \mathbb{Z}_{ét}, \mathbb{Z}^c(-3)) &= 0. \end{aligned}$$

We note that, as expected,

$$\zeta(\operatorname{Spec} \mathbb{Z}, -3) = \zeta(-3) = -\frac{B_4}{4} = \frac{1}{120} = \frac{|H^0(\operatorname{Spec} \mathbb{Z}_{ét}, \mathbb{Z}^c(-3))|}{|H^{-1}(\operatorname{Spec} \mathbb{Z}_{ét}, \mathbb{Z}^c(-3))|}.$$

The localization gives

$$\begin{aligned} H^{-1}(\operatorname{Spec} \mathbb{Z}[1/2]_{ét}, \mathbb{Z}^c(-3)) &\cong H^{-1}(\operatorname{Spec} \mathbb{Z}_{ét}, \mathbb{Z}^c(-3)) \cong \mathbb{Z}/240\mathbb{Z}, \\ H^0(\operatorname{Spec} \mathbb{Z}[1/2]_{ét}, \mathbb{Z}^c(-3)) &\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}, \\ H^1(\operatorname{Spec} \mathbb{Z}[1/2]_{ét}, \mathbb{Z}^c(-3)) &= H^1(\operatorname{Spec} \mathbb{Z}_{ét}, \mathbb{Z}^c(-3)) = 0. \end{aligned}$$

Arithmetically, this corresponds to the fact that the zeta function of $\operatorname{Spec} \mathbb{Z}[1/2]$ has the same Euler product as $\zeta(s)$, with the factor $\frac{1}{1-2^{-s}}$ removed. Therefore, when $s = -3$, the zeta-value should be corrected by $2^3 - 1 = 7$.

For $\mathbb{A}^1_{\mathbb{F}_p}$, we now have

$$H^i(\mathbb{A}^1_{\mathbb{F}_p, ét}, \mathbb{Z}^c(n)) \cong H^{i+2}(\operatorname{Spec} \mathbb{F}_{p, ét}, \mathbb{Z}^c(n-1)) \cong \begin{cases} \mathbb{Z}/(p^{1-n} - 1)\mathbb{Z}, & i = -1, \\ 0, & i \neq -1. \end{cases}$$

In particular, the motivic cohomology of $\mathbb{A}^1_{\mathbb{F}_p}$ is concentrated in

$$H^{-1}(\mathbb{A}^1_{\mathbb{F}_p,\acute{e}t}, \mathbb{Z}^c(-3)) \cong \mathbb{Z}/(p^4-1)\mathbb{Z}.$$

Consider the normalization of X, given by $X' = \operatorname{Spec}\mathbb{Z}[1/2] \sqcup \mathbb{A}^1_{\mathbb{F}_p}$:

$$\begin{array}{ccc} Z' & \hookrightarrow & X' \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & X \end{array}$$

Here $Z = \{\mathfrak{p}\}$, $Z' = \{\mathfrak{P}, \mathfrak{P}'\}$, and

$$\begin{aligned}
\mathfrak{p} &:= \{(a,f) \in \mathbb{Z}[1/2] \times \mathbb{F}_p[t] \mid a \equiv f(0) \equiv 0 \pmod p\},\\
\mathfrak{P} &:= \{(a,f) \in \mathbb{Z}[1/2] \times \mathbb{F}_p[t] \mid a \equiv 0 \pmod p\},\\
\mathfrak{P}' &:= \{(a,f) \in \mathbb{Z}[1/2] \times \mathbb{F}_p[t] \mid f(0) \equiv 0 \pmod p\}.
\end{aligned}$$

The canonical morphism $X' \to X$ induces an isomorphism

$$X' \setminus Z' \cong X \setminus Z \cong (\operatorname{Spec}\mathbb{Z} \setminus \{(2),(p)\}) \sqcup (\operatorname{Spec}\mathbb{F}_p[t] \setminus (t)).$$

We calculate via localizations that

$$\begin{aligned}
H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(-3)) &\cong H^{-1}((X\setminus Z)_{\acute{e}t}, \mathbb{Z}^c(-3)) \cong \mathbb{Z}/240\mathbb{Z} \oplus \mathbb{Z}/(p^4-1)\mathbb{Z},\\
H^0(X_{\acute{e}t}, \mathbb{Z}^c(-3)) &\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/(p^3-1)\mathbb{Z},\\
H^1(X_{\acute{e}t}, \mathbb{Z}^c(-3)) &= 0.
\end{aligned}$$

Consequently,

$$\frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(-3))|}{|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(-3))| \cdot |H^1(X_{\acute{e}t}, \mathbb{Z}^c(-3))|} = \frac{7}{120}\,\frac{p^3-1}{p^4-1}.$$

At the level of zeta-functions,

$$\begin{aligned}
\zeta(X,s) = \zeta(Z,s)\,\zeta(X\setminus Z,s) &= \frac{\zeta(Z,s)}{\zeta(Z',s)}\,\zeta(X',s)\\
&= \frac{1}{\zeta(\operatorname{Spec}\mathbb{F}_p,s)}\,\zeta(\operatorname{Spec}\mathbb{Z}[1/2],s)\,\zeta(\mathbb{A}^1_{\mathbb{F}_p},s)\\
&= (1-p^{-s})\,(1-2^{-s})\,\zeta(s)\,\frac{1}{1-p^{1-s}}.
\end{aligned}$$

In particular, substituting $s=-3$, we get

$$\zeta(X,-3) = -\frac{7}{120}\,\frac{p^3-1}{p^4-1}.$$

References

[1] A. A. Beilinson. Higher regulators and values of L-functions. In *Current problems in mathematics, Vol. 24*, Itogi Nauki i Tekhniki, pages 181–238. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984.

[2] A. Beshenov. Weil-étale cohomology and duality for arithmetic schemes in negative weights, 2021. arXiv preprint 2012.11034, submitted for publication.

[3] A. Beshenov. Weil-étale cohomology and zeta-values of arithmetic schemes at negative integers, 2021. arXiv preprint 2102.12114, submitted for publication.

[4] S. Bloch. Algebraic cycles and higher K-theory. *Adv. in Math.*, 61(3):267–304, 1986.

[5] A. Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974.

[6] A. Borel. Cohomologie de SL_n et valeurs de fonctions zeta aux points entiers. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 4(4):613–636, 1977.

[7] A. I. Borevich and I. R. Shafarevich. *Number theory.* Translated from the Russian by Newcomb Greenleaf. Pure and Applied Mathematics, Vol. 20. Academic Press, New York-London, 1966.

[8] M. Flach and B. Morin. Weil-étale cohomology and zeta-values of proper regular arithmetic schemes. *Doc. Math.*, 23:1425–1560, 2018.

[9] M. Flach and B. Morin. Deninger's conjectures and Weil-Arakelov cohomology. *Münster J. Math.*, 13(2):519–540, 2020.

[10] M. Flach and B. Morin. Compatibility of special value conjectures with the functional equation of zeta functions. *Doc. Math.*, 26:1633–1677, 2021.

[11] T. Geisser. Motivic cohomology over Dedekind rings. *Math. Z.*, 248(4):773–794, 2004.

[12] T. Geisser. Motivic cohomology, K-theory and topological cyclic homology. In *Handbook of K-theory. Vol. 1, 2*, pages 193–234. Springer, Berlin, 2005.

[13] T. Geisser. Duality via cycle complexes. *Ann. of Math. (2)*, 172(2):1095–1126, 2010.

[14] T. Geisser. On the structure of étale motivic cohomology. *J. Pure Appl. Algebra*, 221(7):1614–1628, 2017.

[15] T. Geisser and A. Schmidt. Poitou-Tate duality for arithmetic schemes. *Compos. Math.*, 154(9):2020–2044, 2018.

[16] A. B. Goncharov. Regulators. In *Handbook of K-theory. Vol. 1, 2*, pages 295–349. Springer, Berlin, 2005.

[17] A. Huber and G. Kings. Bloch-Kato conjecture and Main Conjecture of Iwasawa theory for Dirichlet characters. *Duke Math. J.*, 119(3):393–464, 2003.

[18] W. E. Jenner. On zeta functions of number fields. *Duke Math. J.*, 36:669–671, 1969.

[19] B. W. Jordan and B. Poonen. The analytic class number formula for 1-dimensional affine schemes. *Bull. Lond. Math. Soc.*, 52(5):793–806, 2020.

[20] B. Kahn. Some finiteness results for étale cohomology. *J. Number Theory*, 99(1):57–73, 2003.

[21] B. Kahn. Algebraic K-theory, algebraic cycles and arithmetic geometry. In *Handbook of K-theory. Vol. 1, 2*, pages 351–428. Springer, Berlin, 2005.

[22] N. M. Katz. Review of ℓ-adic cohomology. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 21–30. Amer. Math. Soc., Providence, RI, 1994.

[23] M. Kerr, J. D. Lewis, and S. Müller-Stach. The Abel-Jacobi map for higher Chow groups. *Compos. Math.*, 142(2):374–396, 2006.

[24] F. F. Knudsen and D. Mumford. The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div". *Math. Scand.*, 39(1):19–55, 1976.

[25] M. Kolster. K-theory and arithmetic. In *Contemporary developments in algebraic K-theory*, ICTP Lect. Notes, XV, pages 191–258. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[26] M. Kolster and J. W. Sands. Annihilation of motivic cohomology groups in cyclic 2-extensions. *Ann. Sci. Math. Québec*, 32(2):175–187, 2008.

[27] S. Lichtenbaum. Values of zeta-functions, étale cohomology, and algebraic K-theory. In *Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 489–501. Lecture Notes in Math., Vol. 342, 1973.

[28] S. Lichtenbaum. The Weil-étale topology on schemes over finite fields. *Compos. Math.*, 141(3):689–702, 2005.

[29] S. Lichtenbaum. Euler characteristics and special values of zeta-functions. In *Motives and algebraic cycles*, volume 56 of *Fields Inst. Commun.*, pages 249–255. Amer. Math. Soc., Providence, RI, 2009.

[30] S. Lichtenbaum. The Weil-étale topology for number rings. *Ann. of Math. (2)*, 170(2):657–683, 2009.

[31] S. Lichtenbaum. Special values of zeta functions of schemes, 2021. preprint.

[32] J. S. Milne. *Arithmetic duality theorems*. BookSurge, LLC, Charleston, SC, second edition, 2006.

[33] B. Morin. Topological Hochschild homology and zeta-values, 2021. preprint.

[34] J. Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[35] D. Quillen. On the cohomology and K-theory of the general linear groups over a finite field. *Ann. of Math. (2)*, 96:552–586, 1972.

[36] J.-P. Serre. Zeta and L functions. In *Arithmetical Algebraic Geometry (Proc. Conf. Purdue*, pages 82–92. Harper & Row, New York, 1965.

[37] C. Soulé. K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale. *Invent. Math.*, 55(3):251–295, 1979.

[38] C. Weibel. Algebraic K-theory of rings of integers in local and global fields. In *Handbook of K-theory. Vol. 1, 2*, pages 139–190. Springer, Berlin, 2005.

Alexey Beshenov
E-mail: `cadadr@gmail.com`