WEIL-ÉTALE COHOMOLOGY AND DUALITY FOR ARITHMETIC SCHEMES IN NEGATIVE WEIGHTS

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In memory of Bas Edixhoven

Abstract

Flach and Morin [8] constructed Weil-étale cohomology $H^i_{W,c}(X, \mathbb{Z}(n))$ for a proper, regular arithmetic scheme X (i.e. separated and of finite type over $\operatorname{Spec} \mathbb{Z}$) and $n \in \mathbb{Z}$. In the case when $n < 0$, we generalize their construction to an arbitrary arithmetic scheme X, thus removing the proper and regular assumption. The construction uses étale motivic cohomology groups $H^i(X_{ét}, \mathbb{Z}^c(n))$, defined via Bloch cycle complexes [4], which were studied by Geisser [14] in the context of arithmetic duality theorems. It assumes their finite generation for $n < 0$. We give a class of X for which finite generation is known, and hence $H^i_{W,c}(X, \mathbb{Z}(n))$ is defined unconditionally.

1 Introduction

Lichtenbaum, in a series of papers [24, 25, 26], has envisioned a new cohomology theory for schemes, known as **Weil-étale cohomology**. The case of varieties over finite fields $X/\mathbb{F}_q$ was further studied by Geisser [10, 12, 13]. Morin defined in [32] Weil-étale cohomology with compact support $H^i_{W,c}(X, \mathbb{Z})$ for $X \to \operatorname{Spec} \mathbb{Z}$ separated, of finite type, proper, and regular. This construction was further generalized by Flach and Morin in [8] to the groups $H^i_{W,c}(X, \mathbb{Z}(n))$ with arbitrary weights $n \in \mathbb{Z}$, under the same assumptions on X.

The aim of this paper is to remove the assumption that X is proper and regular and, following the ideas of [8], to construct the groups $H^i_{W,c}(X, \mathbb{Z}(n))$ for any X separated and of finite type over $\operatorname{Spec} \mathbb{Z}$ for the case of strictly negative weights $n < 0$.

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As Flach and Morin already suggest in [8, Remark 3.11], we rework all their constructions in terms of $\mathbb{Z}^c(n)$, which is a variant of Bloch's cycle complexes [4, 11], considered by Geisser in [14] in the context of arithmetic duality theorems.

In a forthcoming paper we apply the results of this text to relate the cohomology groups $H^i_{W,c}(X, \mathbb{Z}(n))$ to the special value of the zeta function $\zeta(X, s)$ at $s = n < 0$.

Notation and conventions

Arithmetic schemes. In this work, an **arithmetic scheme** is a scheme X that is separated and of finite type over $\operatorname{Spec} \mathbb{Z}$.

Abelian groups. Let A be an abelian group. For $m \geq 1$ we denote by ${}_mA$ its m-torsion subgroup, and by A_m the quotient A/mA:

$$0 \to {}_mA \to A \xrightarrow{\times m} A \to A_m \to 0.$$

We denote by A_{div} (resp. A_{tor}) the maximal divisible subgroup (resp. maximal torsion subgroup), and by A_{cotor} the quotient A/A_{tor} (following the notation in [8]).

We say that A is of **cofinite type** if it is $\mathbb{Q}/\mathbb{Z}$-dual to a finitely generated abelian group: $A = \operatorname{Hom}(B, \mathbb{Q}/\mathbb{Z})$ for a finitely generated B.

Complexes. All our constructions take place in the derived category of abelian groups $\mathbf{D}(\mathbb{Z})$. For our purposes, we introduce the following terminology. Recall first that a complex of abelian groups $A^\bullet$ is **perfect** if it is bounded (i.e. $H^i(A^\bullet) = 0$ for $|i| \gg 0$), and $H^i(A^\bullet)$ are finitely generated abelian groups.

Definition 1.1. A complex of abelian groups $A^\bullet$ is **almost perfect** if the cohomology groups $H^i(A^\bullet)$ are finitely generated, and bounded, except for possible finite 2-torsion in arbitrarily high degree. That is, $H^i(A^\bullet) = 0$ for $i \ll 0$ and $H^i(A^\bullet)$ is finite 2-torsion for $i \gg 0$.

A complex of abelian groups $A^\bullet$ is of **cofinite type** if the cohomology groups $H^i(A^\bullet)$ are of cofinite type and bounded.

A complex of abelian groups $A^\bullet$ is **almost of cofinite type** if the cohomology groups $H^i(A^\bullet)$ are of cofinite type and bounded, except for possible finite 2-torsion in arbitrarily high degree.

This terminology is ad hoc and was invented for this text, since such complexes will appear frequently. Some basic observations about almost perfect and almost cofinite type complexes are collected in Appendix A. We note that this finite 2-torsion in arbitrarily high degrees could be removed by working with the Artin–Verdier topology $\overline{X}_{ét}$ instead of the usual étale topology $X_{ét}$. The general construction and basic properties of $\overline{X}_{ét}$ are treated in [8, Appendix A], but only for a *proper and regular* arithmetic scheme X. Our methods circumvent this restriction at the cost of some technical hurdles with 2-torsion.

Étale cohomology. For an arithmetic scheme X and a complex of étale sheaves $\mathcal{F}^\bullet$, we denote by

$$R\Gamma(X_{ét}, \mathcal{F}^\bullet) \text{ (resp. } R\Gamma_c(X_{ét}, \mathcal{F}^\bullet),\ R\widehat{\Gamma}_c(X_{ét}, \mathcal{F}^\bullet))$$

the complex that computes the corresponding cohomology, resp. cohomology with compact support, and modified cohomology with compact support. For the convenience of the reader, we review the definitions in Appendix B. The purpose of $R\widehat{\Gamma}_c(X_{ét}, \mathcal{F}^\bullet)$ is to take care of real places $X(\mathbb{R})$. There exists a canonical projection $R\widehat{\Gamma}_c(X_{ét}, \mathcal{F}^\bullet) \to R\Gamma_c(X_{ét}, \mathcal{F}^\bullet)$, which is an isomorphism if $X(\mathbb{R}) = \emptyset$.

***G*-equivariant sheaves and their cohomology.** Let $\mathcal{X}$ be a topological space with an action of a discrete group G. A ***G*-equivariant sheaf** $\mathcal{F}$ on $\mathcal{X}$ can be defined as an espace étalé $\pi\colon E \to \mathcal{X}$ with a G-action on E such that the projection π is G-equivariant (see e.g. [28, §II.6 + pp. 594]). We denote by $\mathbf{Sh}(G, \mathcal{X})$ the corresponding category.

The equivariant global sections are defined by

$$\Gamma(G, \mathcal{X}, \mathcal{F}) = \mathcal{F}(\mathcal{X})^G,$$

with G acting on $\mathcal{F}(\mathcal{X}) = \{s\colon \mathcal{X} \to E \mid \pi \circ s = id_{\mathcal{X}}\}$ via $(g \cdot s)(x) = g \cdot s(g^{-1} \cdot x)$. The corresponding ***G*-equivariant cohomology** is given by the right derived functors of $\Gamma(G, \mathcal{X}, -)$.

More details on G-equivariant sheaves can be found in [31, Chapitre 2]. For our modest purposes, it suffices to know that any G-module A gives rise to the corresponding abelian G-equivariant constant sheaf. The latter corresponds to the espace étalé $\mathcal{X} \times A \to \mathcal{X}$, where A is endowed with the discrete topology.

$G_\mathbb{R}$-equivariant cohomology of $X(\mathbb{C})$. Given an arithmetic scheme X, we denote by $X(\mathbb{C})$ the set of complex points of X endowed with the analytic topology. It carries the natural action of the Galois group $G_\mathbb{R} := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

We consider the $G_\mathbb{R}$-modules

$$\mathbb{Z}(n) := (2\pi i)^n\,\mathbb{Z}, \quad \mathbb{Q}(n) := (2\pi i)^n\,\mathbb{Q}, \quad \mathbb{Q}/\mathbb{Z}(n) := \mathbb{Q}(n)/\mathbb{Z}(n)$$

as constant $G_\mathbb{R}$-equivariant sheaves on $X(\mathbb{C})$.

Then $R\Gamma_c(X(\mathbb{C}), A(n))$ for $A = \mathbb{Z}, \mathbb{Q}, \mathbb{Q}/\mathbb{Z}$ (the complex that computes singular cohomology with compact support of $X(\mathbb{C})$ with coefficients in $A(n)$) is a complex of $G_\mathbb{R}$-modules, and we can further take the group cohomology (resp. Tate cohomology):

$$R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), A(n)) := R\Gamma(G_\mathbb{R}, R\Gamma_c(X(\mathbb{C}), A(n))),$$
$$R\widehat{\Gamma}_c(G_\mathbb{R}, X(\mathbb{C}), A(n)) := R\widehat{\Gamma}(G_\mathbb{R}, R\Gamma_c(X(\mathbb{C}), A(n))).$$

By definition, this is the **$G_\mathbb{R}$-equivariant cohomology** (resp. **$G_\mathbb{R}$-equivariant Tate cohomology**) **with compact support** of $X(\mathbb{C})$ with coefficients in $A(n)$.

Motivic cohomology $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$. Our construction is based on motivic cohomology defined in terms of complexes of sheaves $\mathbb{Z}^c(n)$ on $X_{\acute{e}t}$. The definition goes back to Bloch [4]; see [11] for a survey. We follow the notation of [14] for $\mathbb{Z}^c(n)$.

For $i \ge 0$ we consider the algebraic simplex

$$\Delta^i = \operatorname{Spec} \mathbb{Z}[t_0, \ldots, t_i]/(\sum_i t_i - 1).$$

We fix a non-positive weight $n \le 0$. Let $z_n(X, i)$ be the free abelian group generated by the closed integral subschemes $Z \subset X \times \Delta^i$ of dimension $n+i$ that intersect the faces properly. Then $z_n(X, \bullet)$ is a (homological) complex of abelian groups whose differentials are given by the alternating sum of intersections with the faces. We consider the (cohomological) complex of étale sheaves

$$\mathbb{Z}^c(n) := z_n(\text{\textvisiblespace}, -\bullet)[2n].$$

The boundedness from below of $\mathbb{Z}^c(n)$ is not known in general; it is a variant of the Beilinson–Soulé vanishing conjecture. To work unconditionally with the derived functors, we use K-injective resolutions [36, 34] (resp. K-flat resolutions for the derived tensor products).

To avoid any confusion, we use cohomological numbering for all complexes in this paper, so we set

$$H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) := H^i(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n))).$$

([14] uses homological numbering.)

Assumptions

Weights. In this paper, n normally denotes a strictly negative integer, which will be the weight in the cohomology groups $H^i_{W,c}(X, \mathbb{Z}(n))$. The results in §3 on cohomology of $X(\mathbb{C})$ apply for any weight $n \in \mathbb{Z}$; all other results regarding cohomology groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$, $H^i_{fg}(X, \mathbb{Z}(n))$, $H^i_{W,c}(X, \mathbb{Z}(n))$ apply for $n < 0$.

Finite generation conjecture. Our construction of the Weil-étale cohomology groups $H^i_{W,c}(X, \mathbb{Z}(n))$ uses the following assumption.

CONJECTURE 1.2. $\mathbf{L}^c(X_{ét}, n)$: *for an arithmetic scheme X and $n < 0$, the cohomology groups $H^i(X_{ét}, \mathbb{Z}^c(n))$ are finitely generated for all $i \in \mathbb{Z}$.*

See Proposition 8.3 for consistency of $\mathbf{L}^c(X_{ét}, n)$ with other conjectures that appear in the literature. We refer to §8 for the cases where the conjecture is known.

Main results

Here we state the main results of this paper that are needed for the construction of Weil-étale cohomology. One of our main objects is the following complex of abelian sheaves $\mathbb{Z}(n)$ on $X_{ét}$.

DEFINITION 1.3 ([8, §3.1], [10, §7]). Let X be an arithmetic scheme and $n < 0$. For a prime p, consider the localization $X[1/p]$, and let μ_{p^r} be the sheaf of p^r-th roots of unity on $X[1/p]$. We define the twist of μ_{p^r} by n as

$$\mu_{p^r}^{\otimes n} = \underline{\mathrm{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes(-n)}, \mathbb{Z}/p^r\mathbb{Z}).$$

Now $\mathbb{Z}(n)$ is the complex of sheaves on $X_{ét}$ given by

$$\mathbb{Z}(n) = \mathbb{Q}/\mathbb{Z}(n)[-1], \quad \text{where } \mathbb{Q}/\mathbb{Z}(n) = \bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n},$$

and j_p is the canonical open immersion $X[1/p] \to X$.

The above sheaves $\mathbb{Z}(n)$ should not be confused with cycle complexes; the latter are $\mathbb{Z}^c(n)$ in the context of this paper. In §2 we prove the following arithmetic duality theorem relating the two.

Theorem I. *Assuming Conjecture $\mathbf{L}^c(X_{ét}, n)$, for $n < 0$, there is a quasi-isomorphism*

$$R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n)) \xrightarrow{\cong} R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]).$$

The second result is related to the following morphism of complexes.

DEFINITION 1.4. We define $v_\infty^*\colon R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$ as the morphism in the derived category $\mathbf{D}(\mathbb{Z})$ induced by the comparison of étale and analytic topology

$$\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n)) \to \Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \alpha^*\mathbb{Q}/\mathbb{Z}(n)) \cong \Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$$

(see Proposition B.5 and 6.1). Then we let $u_\infty^*\colon R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ be the composition

$$\begin{aligned} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) := R\Gamma_c(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n))[-1] \xrightarrow{v_\infty^*[-1]} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))[-1] \\ \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \end{aligned}$$

where the last arrow is induced by $\mathbb{Q}/\mathbb{Z}(n)[-1] \to \mathbb{Z}(n)$, which comes from the distinguished triangle of constant $G_\mathbb{R}$-equivariant sheaves $\mathbb{Z}(n) \to \mathbb{Q}(n) \to \mathbb{Q}/\mathbb{Z}(n) \to \mathbb{Z}(n)[1]$.

Theorem II. *Assuming Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, for $n < 0$, the morphism $u_\infty^*\colon R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ is torsion, i.e. there exists a nonzero integer m such that $mu_\infty^* = 0$.*

Outline of the paper

Here we describe the structure of this paper, as well as our construction of the Weil-étale complexes $R\Gamma_{W,c}(X, \mathbb{Z}(n))$.

First, §2 is devoted to the proof of Theorem I. Some of its consequences are deduced in §4. Namely, if we assume Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, then $R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is an almost perfect complex, while $R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n))$ is almost of cofinite type in the sense of Definition 1.1. For this, we first make a small digression in §3 to analyze what kind of complexes we obtain for the $G_\mathbb{R}$-equivariant cohomology of $X(\mathbb{C})$.

Theorem I is used in §5 to define a morphism $\alpha_{X,n}$ in the derived category (see Definition 5.1), and declare $R\Gamma_{fg}(X, \mathbb{Z}(n))$ to be its cone:

$$\begin{aligned} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \\ \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]). \end{aligned}$$

The notation "*fg*" comes from the fact that $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is an almost perfect complex in the sense of Definition 1.1. Thanks to specific properties of the complexes involved, it turns out that $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is defined up to a *unique* isomorphism in the derived category (which is not normally expected from a cone).

Then in §6 we establish Theorem II, and it is used in §7 to define Weil-étale complexes $R\Gamma_{W,c}(X, \mathbb{Z}(n))$. To do this, we deduce from Theorem II that $u_\infty^* \circ \alpha_{X,n} = 0$, which implies that there exists a morphism in the derived category

$$i_\infty^*\colon R\Gamma_{fg}(X, \mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)).$$

We choose a mapping fiber of i_∞^* and call it $R\Gamma_{W,c}(X, \mathbb{Z}(n))$, which turns out to be a perfect complex. The definition of $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ fits in the following commutative diagram with distinguished triangles in the derived category $\mathbf{D}(\mathbb{Z})$:

$$\begin{array}{ccccccc}
 & & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \longrightarrow & 0 & & \\
 & & \text{Dfn. 5.1}\downarrow \alpha_{X,n} & & \downarrow & & \\
 & & R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \xrightarrow[\text{Dfn. 1.4}]{u_\infty^*} & R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) & & \\
 & & \downarrow & & \downarrow id & & \\
R\Gamma_{W,c}(X, \mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X, \mathbb{Z}(n)) & \overset{i_\infty^*}{\dashrightarrow} & R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) & \rightarrow & R\Gamma_{W,c}(X, \mathbb{Z}(n))[1] \\
 & & \downarrow & & \downarrow & & \\
 & & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) & \longrightarrow & 0 & &
\end{array}$$

The resulting complex is the same as defined in [8] if X is proper and regular.

In §8 we consider the cases of X for which Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ is known, and hence our results hold unconditionally, and in §9 we verify that if X is proper and regular, our complex $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is isomorphic to that constructed in [8] by Flach and Morin.

There are two appendices to this paper: Appendix A collects some lemmas from homological algebra, and Appendix B gives an overview of the definitions of étale cohomology with compact support $R\Gamma_c(X_{\acute{e}t}, -)$ and its modified version $R\widehat{\Gamma}_c(X_{\acute{e}t}, -)$.

This work is inspired by [8]. Here is a brief comparison between the notation and assumptions.

this paper	**Flach–Morin**
$X \to \operatorname{Spec} \mathbb{Z}$ separated, of finite type	$X \to \operatorname{Spec} \mathbb{Z}$ proper, regular, equidimensional
$n < 0$	$n \in \mathbb{Z}$
cycle complexes $\mathbb{Z}^c(n)$	cycle complexes $\mathbb{Z}(d-n)[2d]$, $d = \dim X$
$R\Gamma_{fg}(X, \mathbb{Z}(n))$	$R\Gamma_W(\overline{X}, \mathbb{Z}(n))$, up to finite 2-torsion
$R\Gamma_{W,c}(X, \mathbb{Z}(n))$	$R\Gamma_{W,c}(X, \mathbb{Z}(n))$

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[8]. Moreover, the work of Thomas Geisser on arithmetic duality [14] is also crucial for this paper, and his work on Weil-étale cohomology for varieties over finite fields [10, 12, 13] has been of great influence for me. I thank Maxim Mornev for many fruitful mathematical conversations. This paper was edited during my stay at the Center for Research in Mathematics (CIMAT), Guanajuato, Mexico. I am grateful personally to Pedro Luis del Ángel and Xavier Gómez Mont for their hospitality. Finally, I am indebted to the anonymous referee whose sharp and insightful comments on an earlier draft helped to improve the exposition.

2 Proof of Theorem I

At the heart of our constructions is an arithmetic duality theorem for cycle complexes established by Geisser in [14]. The purpose of this section is to deduce Theorem I from Geisser's duality. We would like to obtain a quasi-isomorphism of complexes

$$R\widehat{\Gamma}_c(X_{\acute{e}t},\mathbb{Z}(n)) \xrightarrow{\cong} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}/\mathbb{Z}[-2]).$$

Here $R\widehat{\Gamma}_c(X_{\acute{e}t},\mathbb{Z}(n))$ denotes the modified étale cohomology with compact support, described in Appendix B. We note that [14] uses the notation "$R\Gamma_c$" for our "$R\widehat{\Gamma}_c$", but we take special care to distinguish the two things, since we also need the usual étale cohomology with compact support $R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n))$.

We split our proof of Theorem I into two propositions.

Proposition 2.1. *For any $n<0$ we have a quasi-isomorphism of complexes*

$$R\widehat{\Gamma}_c(X_{\acute{e}t},\mathbb{Z}(n)) \cong \varinjlim_m R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}/m\mathbb{Z}^c(n)),\mathbb{Q}/\mathbb{Z}[-2]). \tag{1}$$

Proof. We unwind our definition of $\mathbb{Z}(n)$ for $n<0$ and reduce everything to the results from [14]. Since $\mathbb{Z}(n) := \bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n}[-1]$, and étale cohomology commutes with filtered colimits of coefficients, it suffices to show that for every prime p and $r \geq 1$ there is a quasi-isomorphism of complexes

$$R\widehat{\Gamma}_c(X_{\acute{e}t},j_{p!}\mu_{p^r}^{\otimes n}[-1]) \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c/p^r(n)),\mathbb{Q}/\mathbb{Z}[-2]). \tag{2}$$

As in Definition 1.3, here j_p denotes the canonical open immersion $j_p\colon X[1/p] \hookrightarrow X$. We further denote by f the structure morphism $X \to \operatorname{Spec}\mathbb{Z}$ and by f_p the structure morphism $X[1/p] \to \operatorname{Spec}\mathbb{Z}[1/p]$:

$$\begin{array}{ccc} X[1/p] & \xhookrightarrow{j_p} & X \\ {\scriptstyle f_p}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}\mathbb{Z}[1/p] & \hookrightarrow & \operatorname{Spec}\mathbb{Z} \end{array}$$

As we are going to change the base scheme, let us write $\mathrm{Hom}_X(-,-)$ for the Hom between sheaves on $X_{\acute{e}t}$ and $\underline{\mathrm{Hom}}_X(-,-)$ for the internal Hom. Instead of $\mathrm{Hom}_{\mathrm{Spec}\,R}$, we will simply write Hom_R.

Applying various results from [9] and [14], we obtain a quasi-isomorphism of complexes of sheaves

$$R\underline{\mathrm{Hom}}_X(j_{p!}\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_X(0)) \cong$$

$$\begin{aligned}
&\cong Rj_{p*}R\underline{\mathrm{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_{X[1/p]}(0)) && \text{[14, Prop. 7.10 c)]}\\
&\cong Rj_{p*}R\underline{\mathrm{Hom}}_{X[1/p]}(f_p^*\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_{X[1/p]}(0)) && \\
&\cong Rj_{p*}Rf_p^!R\underline{\mathrm{Hom}}_{\mathbb{Z}[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_{\mathbb{Z}[1/p]}(0)) && \text{[14, Prop. 7.10 c)]}\\
&\cong Rj_{p*}Rf_p^!R\underline{\mathrm{Hom}}_{\mathbb{Z}[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{G}_{\mathrm{m}}[1]) && \text{[14, Lemma 7.4]}\\
&\cong Rj_{p*}Rf_p^!R\underline{\mathrm{Hom}}_{\mathbb{Z}[1/p]}(\mu_{p^r}^{\otimes n}, \mathbb{G}_{\mathrm{m}})[2] && \\
&\cong Rj_{p*}Rf_p^!\,\mu_{p^r}^{\otimes(1-n)}[2] && \\
&\cong Rj_{p*}Rf_p^!\left(\mathbb{Z}_{\mathbb{Z}[1/p]}/p^r(1-n)\right)[2] \cong Rj_{p*}Rf_p^!\,\mathbb{Z}^c_{\mathbb{Z}[1/p]}/p^r(n) && \text{[9, Thm. 1.2]}\\
&\cong Rj_{p*}\mathbb{Z}^c_{X[1/p]}/p^r(n) && \text{[14, Prop. 7.10 a)]}\\
&\cong Rj_{p*}j_p^*\mathbb{Z}^c_X/p^r(n) \cong \mathbb{Z}^c_X/p^r(n) && \text{[14, Thm. 7.2 a), Prop. 2.3]}
\end{aligned}$$

After applying $R\Gamma(X_{\acute{e}t},-)$, we get a quasi-isomorphism of complexes of abelian groups

$$R\mathrm{Hom}(j_{p!}\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_X(0)) \cong R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c_X/p^r(n)).$$

Now according to the duality [14, Theorem 7.8],

$$R\mathrm{Hom}(j_{p!}\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c(0)) \cong R\mathrm{Hom}(R\widehat{\Gamma}_c(X_{\acute{e}t}, j_{p!}\mu_{p^r}^{\otimes n}[-1]), \mathbb{Q}/\mathbb{Z}[-2]).$$

What we end up with is a quasi-isomorphism

$$R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c/p^r(n)) \cong R\mathrm{Hom}(R\widehat{\Gamma}_c(X_{\acute{e}t}, j_{p!}\mu_{p^r}^{\otimes n}[-1]), \mathbb{Q}/\mathbb{Z}[-2]).$$

The groups $\widehat{H}^i_c(X_{\acute{e}t}, j_{p!}\mu_{p^r}^{\otimes n}[-1])$ are finite (the sheaves $j_{p!}\mu_{p^r}^{\otimes n}$ are constructible), so applying $R\mathrm{Hom}(-,\mathbb{Q}/\mathbb{Z}[-2])$ yields (2). □

To conclude the proof of Theorem I, we identify the complex on the right-hand side of (1). For this, we need Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$.

PROPOSITION 2.2. *Assuming Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, for $n < 0$, there is a quasi-isomorphism*

$$\varinjlim_m R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]) \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]).$$

Proof. Consider short exact sequences

$$0 \to H^i(X_{ét}, \mathbb{Z}^c(n))_m \to H^i(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)) \to {}_mH^{i+1}(X_{ét}, \mathbb{Z}^c(n)) \to 0.$$

If we now take $\operatorname{Hom}(-, \mathbb{Q}/\mathbb{Z})$ and filtered colimits $\varinjlim_m$, we get

$$(3)\quad 0 \to \varinjlim_m \operatorname{Hom}({}_mH^{i+1}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \to \varinjlim_m \operatorname{Hom}(H^i(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \to \varinjlim_m \operatorname{Hom}(H^i(X_{ét}, \mathbb{Z}^c(n))_m, \mathbb{Q}/\mathbb{Z}) \to 0.$$

By Conjecture $\mathbf{L}^c(X_{ét}, n)$, the group $H^{i+1}(X_{ét}, \mathbb{Z}^c(n))$ is finitely generated, and hence the first $\varinjlim_m$ in the short exact sequence (3) vanishes, and we obtain isomorphisms

$$\varinjlim_m \operatorname{Hom}(H^i(X_{ét}, \mathbb{Z}^c(n))_m, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \varinjlim_m \operatorname{Hom}(H^i(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}).$$

It remains to note that the left-hand side is canonically isomorphic to $\operatorname{Hom}(H^i(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z})$, again thanks to the finite generation of $H^i(X_{ét}, \mathbb{Z}^c(n))$, under Conjecture $\mathbf{L}^c(X_{ét}, n)$.

To see this, observe that if A is a finitely generated abelian group, there is a canonical isomorphism

$$\varinjlim_m \operatorname{Hom}(A_m, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z})$$

induced by $A \to A_m$, and then applying the functor $\operatorname{Hom}(-, \mathbb{Q}/\mathbb{Z})$ and $\varinjlim_m$. Since $\mathbb{Q}/\mathbb{Z}$ is a torsion group, any homomorphism $A \to \mathbb{Q}/\mathbb{Z}$ is killed by some m, hence factors through A_m. □

3 $G_\mathbb{R}$-equivariant cohomology of $X(\mathbb{C})$

Lemma 3.1. *Let $A^\bullet$ be a perfect complex of $\mathbb{Z}G_\mathbb{R}$-modules.*

1) *The complex $A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z}$ is of cofinite type.*

2) *$R\Gamma(G_\mathbb{R}, A^\bullet \otimes \mathbb{Q}) \cong (A^\bullet \otimes \mathbb{Q})^{G_\mathbb{R}}$ is a perfect complex of $\mathbb{Q}$-vector spaces, and the complex $R\widehat{\Gamma}(G_\mathbb{R}, A^\bullet \otimes \mathbb{Q})$ is quasi-isomorphic to 0.*

3) *$R\widehat{\Gamma}(G_\mathbb{R}, A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z}) \cong R\widehat{\Gamma}(G_\mathbb{R}, A^\bullet[+1])$, and these complexes have finite 2-torsion cohomology.*

4) *$R\Gamma(G_\mathbb{R}, A^\bullet)$ is almost perfect, and $R\Gamma(G_\mathbb{R}, A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z})$ is almost of cofinite type.*

PROOF. The universal coefficient theorem gives us short exact sequences

$$0 \to H^i(A^\bullet)_m \to H^i(A^\bullet \otimes^{\mathbf{L}} \mathbb{Z}/m\mathbb{Z}) \to {}_mH^{i+1}(A^\bullet) \to 0.$$

The colimit of these over m is

$$0 \to H^i(A^\bullet) \otimes \mathbb{Q}/\mathbb{Z} \to H^i(A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z}) \to H^{i+1}(A^\bullet)_{tor} \to 0.$$

Here $H^i(A^\bullet) \otimes \mathbb{Q}/\mathbb{Z}$ is injective, hence the short exact sequence splits. We see that $H^i(A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z})$ is of cofinite type and vanishes for $|i| \gg 0$, i.e. that $A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z}$ is of cofinite type.

Let us now consider the spectral sequences

$$E_2^{pq} = H^p(G_\mathbb{R}, H^q(A^\bullet \otimes \mathbb{Q})) \Longrightarrow H^{p+q}(G_\mathbb{R}, A^\bullet \otimes \mathbb{Q}), \tag{4}$$

$$E_2^{pq} = \widehat{H}^p(G_\mathbb{R}, H^q(A^\bullet \otimes \mathbb{Q})) \Longrightarrow \widehat{H}^{p+q}(G_\mathbb{R}, A^\bullet \otimes \mathbb{Q}). \tag{5}$$

We recall that $H^p(G_\mathbb{R}, -)$ are 2-torsion groups for $p > 0$. Since $H^q(A^\bullet \otimes \mathbb{Q})$ are $\mathbb{Q}$-vector spaces, it follows that $E_2^{pq} = 0$ for $p > 0$ in (4), and the spectral sequence degenerates. Similarly, the Tate cohomology groups $\widehat{H}^p(G_\mathbb{R}, H^q(A^\bullet \otimes \mathbb{Q}))$ are trivial for *all* p for the same reasons, so that (5) is trivial. This proves part 2).

Part 3) now follows from the distinguished triangle

$$R\widehat{\Gamma}(G_\mathbb{R}, A^\bullet) \to R\widehat{\Gamma}(G_\mathbb{R}, A^\bullet \otimes \mathbb{Q}) \to R\widehat{\Gamma}(G_\mathbb{R}, A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z}) \to R\widehat{\Gamma}(G_\mathbb{R}, A^\bullet)[1].$$

Next, examining the spectral sequence

$$E_2^{pq} = H^p(G_\mathbb{R}, H^q(A^\bullet)) \Longrightarrow H^{p+q}(G_\mathbb{R}, A^\bullet),$$

we see that the groups $H^i(G_\mathbb{R}, A^\bullet)$ are finitely generated, zero for $i \ll 0$, and torsion for $i \gg 0$. The latter is 2-torsion. To see that, let $P_\bullet \twoheadrightarrow \mathbb{Z}$ be the bar-resolution of $\mathbb{Z}$ by free $\mathbb{Z}G_\mathbb{R}$-modules. Consider the morphism of complexes

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & P_3 & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow 0 \\
& \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2-N} & \\
\cdots \longrightarrow & P_3 & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow 0
\end{array}$$

where N denotes the norm map. The proof of [39, Theorem 6.5.8] shows that the above morphism induces multiplication by 2 on $H^i(G_\mathbb{R}, -)$ for $i > 0$, and it is null-homotopic. Since $A^\bullet$ is bounded, we see that the above morphism induces multiplication by 2 on $H^i(G_\mathbb{R}, A^\bullet)$ for $i \gg 0$.

Similarly, analyzing

$$E_2^{pq} = H^p(G_\mathbb{R}, H^q(A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z})) \Longrightarrow H^{p+q}(G_\mathbb{R}, A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z}).$$

we see that $H^i(G_{\mathbb{R}}, A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z})$ are groups of cofinite type. To see that these are finite 2-torsion for $i \gg 0$, consider the triangle

$$R\Gamma(G_{\mathbb{R}}, A^\bullet) \to R\Gamma(G_{\mathbb{R}}, A^\bullet \otimes \mathbb{Q}) \to R\Gamma(G_{\mathbb{R}}, A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z}) \to R\Gamma(G_{\mathbb{R}}, A^\bullet)[1].$$

Here $R\Gamma(G_{\mathbb{R}}, A^\bullet \otimes \mathbb{Q})$ is bounded, and therefore $H^i(G_{\mathbb{R}}, A^\bullet \otimes^{\mathbf{L}} \mathbb{Q}/\mathbb{Z}) \cong H^{i+1}(G_{\mathbb{R}}, A^\bullet)$ for $i \gg 0$. □

PROPOSITION 3.2. *Let X be an arithmetic scheme and $n \in \mathbb{Z}$. Then $X(\mathbb{C})$ has the following types of complexes as its cohomology:*

	$A = \mathbb{Z}$	$A = \mathbb{Q}$	$A = \mathbb{Q}/\mathbb{Z}$
$R\Gamma_c(X(\mathbb{C}), A(n))$	*perfect*$_{/\mathbb{Z}}$	*perfect*$_{/\mathbb{Q}}$	*cofinite type*
$R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), A(n))$	*almost perfect*	*perfect*$_{/\mathbb{Q}}$	*almost cofinite type*
$R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), A(n))$	*finite 2-torsion*	$\cong 0$	*finite 2-torsion*

Moreover, there is an isomorphism

$$\widehat{H}^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \cong H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \quad \textit{for } i \geq 2\dim X - 1. \tag{6}$$

This result is purely topological and holds for any $n \in \mathbb{Z}$, unlike other results of this paper regarding motivic cohomology that are stated for $n < 0$. Here $\mathbb{Z}(n)$, $\mathbb{Q}(n)$, $\mathbb{Q}/\mathbb{Z}(n)$ are the constant $G_{\mathbb{R}}$-equivariant sheaves $(2\pi i)^n\,\mathbb{Z}$, $(2\pi i)^n\,\mathbb{Q}$, $\mathbb{Q}(n)/\mathbb{Z}(n)$ respectively. Their relation to the sheaves $\mathbb{Z}(n)$, $\mathbb{Q}(n)$, $\mathbb{Q}/\mathbb{Z}(n)$ on $X_{ét}$ (Definition 1.3) is given by Proposition 6.1 below.

PROOF. We claim that $H^q_c(X(\mathbb{C}), \mathbb{Z}(n))$ are finitely generated groups, and

$$H^q_c(X(\mathbb{C}), \mathbb{Z}(n)) = 0 \quad \text{for } q \notin [0, 2\dim X - 2]. \tag{7}$$

We may assume $X(\mathbb{C}) \neq \emptyset$. The topological dimension of $X(\mathbb{C})$ satisfies $\dim X = 1 + \dim X_{\mathbb{C}} = 1 + \frac{1}{2}\dim_{top} X(\mathbb{C})$, so that $\dim_{top} X(\mathbb{C}) = 2\dim X - 2$.

If $X(\mathbb{C})$ is smooth, we may assume it is of pure dimension $d = \dim_{top} X(\mathbb{C})$. Then finite generation and (7) follow from the Poincaré duality

$$H^i_c(X(\mathbb{C}), \mathbb{Z}(n)) \cong H_{2d-i}(X(\mathbb{C}), \mathbb{Z}(n)),$$

and the fact that $X(\mathbb{C})$ has the homotopy type of a finite CW-complex by van der Waerden's theorem (see [38] and more recent expositions with more general results in [27, 21]).

In the general case, we use induction on the dimension of $X(\mathbb{C})$. Consider the decomposition $U(\mathbb{C}) \hookrightarrow X(\mathbb{C}) \hookleftarrow Z(\mathbb{C})$, where $Z(\mathbb{C})$ is the singular locus. In the long exact sequence

$$\cdots \to H^q_c(U(\mathbb{C}), \mathbb{Z}(n)) \to H^q_c(X(\mathbb{C}), \mathbb{Z}(n)) \to H^q_c(Z(\mathbb{C}), \mathbb{Z}(n)) \to H^{q+1}_c(U(\mathbb{C}), \mathbb{Z}(n)) \to \cdots$$

the groups $H^q_c(U(\mathbb{C}), \mathbb{Z}(n))$ are finitely generated by the smooth case, and $H^q_c(Z(\mathbb{C}), \mathbb{Z}(n))$ are finitely generated by induction hypothesis. It follows that $H^q_c(X(\mathbb{C}), \mathbb{Z}(n))$ are finitely generated. Similarly we conclude by induction that (7) holds.

The rest of the table is an application of the previous lemma to $R\Gamma_c(X(\mathbb{C}), \mathbb{Z}(n))$.

Finally, (6) follows from the spectral sequences

$$\begin{aligned} \widehat{E}_2^{pq} &= \widehat{H}^p(G_\mathbb{R}, H^q_c(X(\mathbb{C}), \mathbb{Z}(n))) \Longrightarrow \widehat{H}^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)), \\ E_2^{pq} &= H^p(G_\mathbb{R}, H^q_c(X(\mathbb{C}), \mathbb{Z}(n))) \Longrightarrow H^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)), \end{aligned}$$

using (7) and the isomorphism $\widehat{H}^p(G_\mathbb{R}, -) \cong H^p(G_\mathbb{R}, -)$ for $p \geq 1$. □

4 Some consequences of Theorem I

Now we deduce some consequences from the duality Theorem I.

Lemma 4.1. *The canonical morphism $\phi^i\colon \widehat{H}^i_c(X_{ét}, \mathbb{Z}(n)) \to H^i_c(X_{ét}, \mathbb{Z}(n))$ sits in a long exact sequence*

$$\begin{aligned} \cdots \to \widehat{H}^{i-1}_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \to \widehat{H}^i_c(X_{ét}, \mathbb{Z}(n)) \xrightarrow{\phi^i} H^i_c(X_{ét}, \mathbb{Z}(n)) \\ \to \widehat{H}^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \to \cdots \end{aligned}$$

where the groups $\widehat{H}^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ are finite 2-torsion. In particular,

1) *the kernel and cokernel of ϕ^i are finite 2-torsion,*

2) *if $X(\mathbb{R}) = \emptyset$, then $R\widehat{\Gamma}_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) = 0$ and $\widehat{H}^i_c(X_{ét}, \mathbb{Z}(n)) \cong H^i_c(X_{ét}, \mathbb{Z}(n))$.*

Proof. The exact sequence follows from the definition of modified étale cohomology with compact support and Artin's comparison theorem. This is proved in [8, Lemma 6.14]. In particular, the argument shows that $R\widehat{\Gamma}_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \cong R\widehat{\Gamma}(G_\mathbb{R}, v^* Rf_* \mathbb{Z}(n))$ where $v\colon \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{Z}$ and $f\colon X \to \operatorname{Spec} \mathbb{Z}$, and $R\widehat{\Gamma}_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) = 0$ if $X(\mathbb{R}) = \emptyset$.

The fact that $\widehat{H}^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ are finite 2-torsion is a part of Proposition 3.2. □

Proposition 4.2. *Let X be an arithmetic scheme of dimension d satisfying Conjecture $\mathbf{L}^c(X_{ét}, n)$, let $n < 0$.*

1) *If $X(\mathbb{R}) = \emptyset$, then $H^i(X_{ét}, \mathbb{Z}^c(n)) = 0$ for $i > 1$ or $i < -2d$.*

2) *In general, $H^i(X_{ét}, \mathbb{Z}^c(n)) = 0$ for $i < -2d$, and $H^i(X_{ét}, \mathbb{Z}^c(n))$ is a finite 2-torsion group for $i > 1$.*

3) *If $X/\mathbb{F}_q$ is a variety over a finite field, then the groups $H^i(X_{ét}, \mathbb{Z}^c(n))$ are finite for all $i \in \mathbb{Z}$.*

In general, we have the following cohomology for $n < 0$:

groups	***type***	$i \ll 0$		$i \gg 0$	
$H^i(X_{ét}, \mathbb{Z}^c(n))$	*finitely generated*	0	*for $i < -2d$*	*finite 2-torsion*	*for $i > 1$*
$\widehat{H}^i_c(X_{ét}, \mathbb{Z}(n))$	*cofinite*	*finite 2-torsion*	*for $i < 1$*	0	*for $i > 2d+2$*
$H^i_c(X_{ét}, \mathbb{Z}(n))$	*cofinite*	0	*for $i < 1$*	*finite 2-torsion*	*for $i > 2d+2$*

In particular, $R\Gamma(X_{ét}, \mathbb{Z}^c(n))$ is an almost perfect complex, while $R\Gamma_c(X_{ét}, \mathbb{Z}(n))$ is almost of cofinite type in the sense of Definition 1.1.

Proof. If $X(\mathbb{R}) = \emptyset$, then our duality Theorem I gives

$$\mathrm{Hom}(H^{2-i}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \cong \widehat{H}^i_c(X_{ét}, \mathbb{Z}(n)) \overset{X(\mathbb{R})=\emptyset}{\cong} H^i_c(X_{ét}, \mathbb{Z}(n)).$$

We have $H^i_c(X_{ét}, \mathbb{Z}(n)) = 0$ for $i < 1$ by the definition of $\mathbb{Z}(n)$, and $H^i_c(X_{ét}, \mathbb{Z}(n)) = H^{i-1}(X_{ét}, \mathbb{Q}/\mathbb{Z}(n)) = 0$ for $i > 2d+2$ for reasons of ℓ-adic cohomological dimension [1, Exposé X, Théorème 6.2]. This proves part 1) of the proposition.

In part 2), the group $H^i(X_{ét}, \mathbb{Z}^c(n))$ is finite 2-torsion for $i > 1$, thanks to part 1) and Lemma 4.1. Moreover, we have $H^i(X_{ét}, \mathbb{Z}^c(n)) \cong H^i(X_{ét}, \mathbb{Q}^c(n))$ for $i < -2d$ according to [32, Lemma 5.12]. Conjecture $L^c(X_{ét}, n)$ implies that these groups are $\mathbb{Q}$-vector spaces finitely generated over $\mathbb{Z}$, hence trivial.

In part 3), the cohomology groups $H^i(X_{ét}, \mathbb{Z}(n)) = H^{i-1}(X_{ét}, \mathbb{Q}/\mathbb{Z}(n))$ are finite for $n < 0$ by [22, Theorem 3]. □

5 Complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$

The purpose of this section is to define auxiliary complexes $R\Gamma_{fg}(X, \mathbb{Z}(n))$, which are used below in the construction of Weil-étale cohomology.

Definition 5.1. Assuming Conjecture $\mathbf{L}^c(X_{ét}, n)$ and $n<0$, consider the morphism $\alpha_{X,n}$ in the derived category $\mathbf{D}(\mathbb{Z})$ given by the composition

$$\begin{array}{ccc}
R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow{\mathbb{Q}\twoheadrightarrow\mathbb{Q}/\mathbb{Z}} & R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]) \\
 & & \uparrow {\scriptstyle \text{Theorem I}\ \cong} \\
 & & R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n)) \\
 & \searrow^{\alpha_{X,n}} & \downarrow {\scriptstyle \text{proj.}} \\
 & & R\Gamma_c(X_{ét}, \mathbb{Z}(n))
\end{array}$$

Here the first arrow is induced by the canonical projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$, and the last arrow is the canonical projection from the modified cohomology with compact support to the usual cohomology with compact support (see Appendix B).

We define the complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$ as a cone of $\alpha_{X,n}$:

$$\begin{aligned}
R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{ét}, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \\
\to R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]).
\end{aligned}$$

Further, we denote

$$H^i_{fg}(X, \mathbb{Z}(n)) := H^i(R\Gamma_{fg}(X, \mathbb{Z}(n))).$$

Remark 5.2. Under Conjecture $\mathbf{L}^c(X_{ét}, n)$, the groups $H^i_c(X_{ét}, \mathbb{Z}(n))$ for $n<0$ are of cofinite type by Theorem I, while $R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-2])$ is a complex of $\mathbb{Q}$-vector spaces. Therefore, the morphism $\alpha_{X,n}$ is completely determined by the maps between cohomology groups

$$H^i(\alpha_{X,n})\colon \mathrm{Hom}(H^{2-i}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}) \to H^i_c(X_{ét}, \mathbb{Z}(n))$$

—see Lemma A.5.

Remark 5.3. We note that our $R\Gamma_{fg}(X, \mathbb{Z}(n))$ plays the same role as $R\Gamma_W(\overline{X}_{ét}, \mathbb{Z}(n))$ in [8, Definition 3.6]. We use a different notation since Flach and Morin work with the Artin–Verdier topology and their complex $R\Gamma_W(\overline{X}_{ét}, \mathbb{Z}(n))$ is perfect, while our complex can have finite 2-torsion in arbitrarily high degree.

We first note that the definition simplifies when X has no real places.

Proposition 5.4. *Assuming conjecture $\mathbf{L}^c(X_{ét}, n)$, for $n<0$, if $X(\mathbb{R}) = \emptyset$, then*

$$R\Gamma_{fg}(X, \mathbb{Z}(n)) \cong R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]).$$

Proof. In this case $R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n)) \to R\Gamma_c(X_{ét}, \mathbb{Z}(n))$ is the identity morphism, and therefore $\alpha_{X,n}$ sits in the following commutative diagram with distinguished columns:

$$\begin{array}{ccc}
R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow{\mathrm{id}} & R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \\
\downarrow \alpha_{X,n} & & \downarrow \\
R\Gamma_c(X_{ét}, \mathbb{Z}(n)) & \xrightarrow[\text{Theorem I}]{\cong} & R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]) \\
\downarrow & & \downarrow \\
R\Gamma_{fg}(X, \mathbb{Z}(n)) & \overset{\cong}{\dashrightarrow} & R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]) \\
\downarrow & & \downarrow \\
R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) & \xrightarrow{\mathrm{id}} & R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-1])
\end{array}$$

Here the first column is our definition of $R\Gamma_{fg}(X, \mathbb{Z}(n))$, and the second column is induced by the distinguished triangle $\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Z}[1]$. □

Proposition 5.5. *Assuming Conjecture $\mathbf{L}^c(X_{ét}, n)$, the complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$ for $n < 0$ is almost perfect in the sense of Definition* 1.1*, i.e. its cohomology groups $H^i_{fg}(X, \mathbb{Z}(n))$ are finitely generated, trivial for $i \ll 0$, and* 2*-torsion for $i \gg 0$.*

Proof. By the definition of $R\Gamma_{fg}(X, \mathbb{Z}(n))$, there are short exact sequences

$$0 \to \operatorname{coker} H^i(\alpha_{X,n}) \to H^i_{fg}(X, \mathbb{Z}(n)) \to \ker H^{i+1}(\alpha_{X,n}) \to 0.$$

The morphism $\alpha_{X,n}$ is given at the level of cohomology by

$$H^i(\alpha_{X,n})\colon \mathrm{Hom}(H^{2-i}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}) \xrightarrow{\psi^i} \mathrm{Hom}(H^{2-i}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \widehat{H}^i_c(X_{ét}, \mathbb{Z}(n)) \xrightarrow{\phi^i} H^i_c(X_{ét}, \mathbb{Z}(n))$$

where $H^{2-i}(X_{ét}, \mathbb{Z}^c(n))$ is a finitely generated abelian group according to $\mathbf{L}^c(X_{ét}, n)$. We consider the ker-coker exact sequence (ignoring the isomorphism in the middle)

$$0 \to \underbrace{\mathrm{Hom}(H^{2-i}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Z})}_{\cong \ker \psi^i} \to \ker H^i(\alpha_{X,n}) \to \ker \phi^i \to \underbrace{\mathrm{Hom}(H^{2-i}(X_{ét}, \mathbb{Z}^c(n))_{tor}, \mathbb{Q}/\mathbb{Z})}_{\cong \operatorname{coker} \psi^i} \to \operatorname{coker} H^i(\alpha_{X,n}) \to \operatorname{coker} \phi^i \to 0.$$

Here $\ker \phi^i$ and $\operatorname{coker} \phi^i$ are finite 2-torsion according to Lemma 4.1, and $H^\bullet(X_{ét}, \mathbb{Z}^c(n))$ are finitely generated by $\mathbf{L}^c(X_{ét}, n)$. This establishes finite generation of $\ker H^{i+1}(\alpha_{X,n})$ and $\operatorname{coker} H^i(\alpha_{X,n})$, and hence of $H^i_{fg}(X, \mathbb{Z}(n))$.

From the description of cohomology groups in Proposition 4.2, for $i \ll 0$ we have $\mathrm{Hom}(H^{2-i}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}) = H^i_c(X_{ét}, \mathbb{Z}(n)) = 0$, and hence $H^i_{fg}(X, \mathbb{Z}(n)) = 0$. On the other hand, for $i \gg 0$ we have $\mathrm{Hom}(H^{2-i}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}) = 0$, so that $H^i_{fg}(X, \mathbb{Z}(n)) \cong H^i_c(X_{ét}, \mathbb{Z}(n))$ is a finite 2-torsion group. □

PROPOSITION 5.6. *Assuming conjecture $\mathbf{L}^c(X_{\acute{e}t},n)$, for $n<0$, the complex $R\Gamma_{fg}(X,\mathbb{Z}(n))$ is defined up to a unique isomorphism in the derived category $\mathbf{D}(\mathbb{Z})$.*

PROOF. The complex $R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2])$ consists of $\mathbb{Q}$-vector spaces, and $R\Gamma_{fg}(X,\mathbb{Z}(n))$ is almost perfect, so we are in the situation of Corollary A.3. □

PROPOSITION 5.7. *Suppose that Conjecture $\mathbf{L}^c(X_{\acute{e}t},n)$ holds for $n<0$ and consider the distinguished triangle defining $R\Gamma_{fg}(X,\mathbb{Z}(n))$:*

$$R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) \xrightarrow{f} R\Gamma_{fg}(X,\mathbb{Z}(n)) \xrightarrow{g} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]).$$

1) *The morphism g induces an isomorphism*

$$g\otimes\mathbb{Q}\colon R\Gamma_{fg}(X,\mathbb{Z}(n))\otimes\mathbb{Q} \xrightarrow{\cong} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]).$$

2) *For each $m\geq 1$ the morphism f induces an isomorphism*

$$f\otimes\mathbb{Z}/m\mathbb{Z}\colon R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n))\otimes^{\mathbf{L}}\mathbb{Z}/m\mathbb{Z} \xrightarrow{\cong} R\Gamma_{fg}(X,\mathbb{Z}(n))\otimes^{\mathbf{L}}\mathbb{Z}/m\mathbb{Z}.$$

3) *For any prime ℓ the morphism f induces an isomorphism*

$$\varprojlim_r H^i_c(X_{\acute{e}t},\mathbb{Z}/\ell^r(n)) \cong H^i_{fg}(X,\mathbb{Z}(n))\otimes\mathbb{Z}_\ell.$$

PROOF. The groups $H^i_c(X_{\acute{e}t},\mathbb{Z}(n))$ are all torsion, and therefore $R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n))\otimes\mathbb{Q}\cong 0$ in the derived category. Similarly, the complexes of $\mathbb{Q}$-vector spaces $R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[\cdots])$ are killed by tensoring with $\mathbb{Z}/m\mathbb{Z}$. This proves 1) and 2).

Now 2) implies 3): by the finite generation of $H^i_{fg}(X,\mathbb{Z}(n))$, we have

$$\varprojlim_r H^i_c(X_{\acute{e}t},\mathbb{Z}/\ell^r(n)) \overset{2)}{\cong} \varprojlim_r H^i_{fg}(X,\mathbb{Z}/\ell^r(n)) \cong \varprojlim_r H^i_{fg}(X,\mathbb{Z}(n))/\ell^r \cong H^i_{fg}(X,\mathbb{Z}(n))\otimes\mathbb{Z}_\ell.$$

□

The groups $H^i_{fg}(X,\mathbb{Z}(n))$ provide an integral model for ℓ-adic cohomology in the following sense (see also [10, §8]).

COROLLARY 5.8. *Let X be an arithmetic scheme satisfying Conjecture $\mathbf{L}^c(X_{\acute{e}t},n)$ for $n<0$. Then*

$$H^i_{fg}(X,\mathbb{Z}(n))\otimes\mathbb{Z}_\ell \cong H^i_c(X[1/\ell]_{\acute{e}t},\mathbb{Z}_\ell(n)),$$

where the right-hand side denotes ℓ-adic cohomology with compact support.

PROOF. We have $\mathbb{Z}(n)/\ell^r \cong j_{\ell!}\mu_m^{\otimes n}$. Now by part 3) of the previous proposition,

$$H^i_{fg}(X,\mathbb{Z}(n))\otimes\mathbb{Z}_\ell \cong \varprojlim_r H^i_c(X_{\acute{e}t},j_{\ell!}\mu_{\ell^r}^{\otimes n}) \cong \varprojlim_r H^i_c(X[1/\ell]_{\acute{e}t},\mu_{\ell^r}^{\otimes n}) \overset{\mathrm{dfn}}{=} H^i_c(X[1/\ell]_{\acute{e}t},\mathbb{Z}_\ell(n)).$$

□

6 Proof of Theorem II

The aim of this section is to prove Theorem II. We recall that it states that the morphism of complexes u_∞^*, defined as the composition

$$\begin{array}{ccc} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \overset{u_\infty^*}{\dashrightarrow} & R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \\ \| & & \uparrow \\ R\Gamma_c(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n))[-1] & \xrightarrow{v_\infty^*[-1]} & R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))[-1] \end{array}$$

is torsion. Here $v_\infty^*\colon R\Gamma_c(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$ is induced by the comparison functor $\alpha^*\colon \mathbf{Sh}(X_{\acute{e}t}) \to \mathbf{Sh}(G_\mathbb{R}, X(\mathbb{C}))$, as explained in Proposition B.5. We first ensure that α^* identifies the sheaf $\mathbb{Q}/\mathbb{Z}(n)$ on $X_{\acute{e}t}$ from Definition 1.3 with the $G_\mathbb{R}$-equivariant sheaf $\mathbb{Q}/\mathbb{Z}(n) := \frac{(2\pi i)^n\,\mathbb{Q}}{(2\pi i)^n\,\mathbb{Z}}$ on $X(\mathbb{C})$.

Proposition 6.1. *For the sheaf $\mathbb{Q}/\mathbb{Z}(n)$ on $X_{\acute{e}t}$ we have an isomorphism of $G_\mathbb{R}$-equivariant constant sheaves on $X(\mathbb{C})$*

$$\alpha^*\mathbb{Q}/\mathbb{Z}(n) \cong \mathbb{Q}/\mathbb{Z}(n).$$

Proof. We first compute that the functor α^* sends the sheaf $\mu_m^{\otimes n}$ on $X_{\acute{e}t}$ to the constant $G_\mathbb{R}$-equivariant sheaf $\frac{(2\pi i)^n\,\mathbb{Z}}{m\,(2\pi i)^n\,\mathbb{Z}}$ on $X(\mathbb{C})$:

$$\begin{aligned} \alpha^*\mu_m^{\otimes n} &\cong \mu_m(\mathbb{C})^{\otimes n} := \underline{\mathrm{Hom}}(\mu_m(\mathbb{C})^{\otimes(-n)}, \mathbb{Z}/m\mathbb{Z}) \\ &\cong \frac{(2\pi i)^n\,\mathbb{Z}}{m\,(2\pi i)^n\,\mathbb{Z}} \end{aligned}$$

—here the first isomorphism comes from the definition of α^* given in Appendix B, and the second isomorphism comes from the corresponding isomorphism of $G_\mathbb{R}$-modules.

Since α^* preserves colimits (Lemma B.4), we have

$$\alpha^*\mathbb{Q}/\mathbb{Z}(n) = \alpha^*\Big(\bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n}\Big) \cong \varinjlim_m \alpha^*\mu_m^{\otimes n} \cong \varinjlim_m \frac{(2\pi i)^n\,\mathbb{Z}}{m\,(2\pi i)^n\,\mathbb{Z}} \cong \frac{(2\pi i)^n\,\mathbb{Q}}{(2\pi i)^n\,\mathbb{Z}}. \qquad \square$$

We proceed with our proof of Theorem II. Our argument follows the proof of [8, Lemma 3.25]. We'll need following result about ℓ-adic cohomology.

Proposition 6.2. *Let X be an arithmetic scheme and $n < 0$. Then for any prime ℓ we have*

$$(H_c^i(X_{\overline{\mathbb{Q}},\acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_\mathbb{Q}})_{div} = 0.$$

Proof. We claim that for a suitable choice of prime $p \neq \ell$,

$$H_c^i(X_{\overline{\mathbb{Q}},\acute{e}t}, \mathbb{Z}_\ell(n)) \cong H_c^i(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Z}_\ell(n)) \quad \text{for all } i.$$

We have $f\colon X \to \operatorname{Spec}\mathbb{Z}$, separated, of finite type. $\mathbb{Z}_\ell(n)$ is a constructible $\mathbb{Z}_\ell$-sheaf on X in the sense of [17, Exposé VI, 1.1.1], and by [ibid., 2.2.2], $R^i f_! \mathbb{Z}_\ell(n)$ is a constructible $\mathbb{Z}_\ell$-sheaf on $\operatorname{Spec}\mathbb{Z}$. Now [ibid., 1.2.6] implies that there exists an open subscheme $U = \operatorname{Spec}\mathbb{Z}_S \subset \operatorname{Spec}\mathbb{Z}$ such that $R^i f_! \mathbb{Z}_\ell(n)$ is a twisted constant sheaf on U. We may take a finite set of primes S such that this holds for all i. Then for $p \notin S$, the proper base change for constructible sheaves [ibid. 2.2.3] applied to the diagram

$$\begin{array}{ccccc} X_{\overline{\mathbb{Q}}} & \longrightarrow & X_U & \longleftarrow & X_{\overline{\mathbb{F}_p}} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f_U} & \llcorner & \downarrow \\ \operatorname{Spec}\overline{\mathbb{Q}} & \xrightarrow[\overline{\eta}]{} & \operatorname{Spec}\mathbb{Z}_S & \xleftarrow[\overline{x}]{} & \operatorname{Spec}\overline{\mathbb{F}}_p \end{array}$$

gives us an isomorphism

$$H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t}, \mathbb{Z}_\ell(n)) \cong (R^i f_{U,!}\mathbb{Z}_\ell(n))_{\overline{\eta}} \cong (R^i f_{U,!}\mathbb{Z}_\ell(n))_{\overline{x}} \cong H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Z}_\ell(n)). \tag{8}$$

We denote by I_p the inertia subgroup of the absolute Galois group $G_{\mathbb{Q}_p}$:

$$1 \to I_p \to G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p} \to 1.$$

The isomorphism (8) is equivariant under the $G_{\mathbb{Q}_p}$-action on the left-hand side and $G_{\mathbb{F}_p}$-action on the right-hand side. We have

$$H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{Q}}} \rightarrowtail H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{Q}_p}/I_p} \cong H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{F}_p}},$$

so it suffices to show that

$$(H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{F}_p}})_{div} = 0.$$

The long exact sequence of $G_{\mathbb{F}_p}$-modules

$$\cdots \to H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Z}_\ell(n)) \to H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Q}_\ell(n)) \to H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \\ \to H^{i+1}_c(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Z}_\ell(n)) \to \cdots$$

induces short exact sequences

$$0 \to H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Z}_\ell(n))_{cotor} \to H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Q}_\ell(n)) \to H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div} \to 0. \tag{9}$$

Here $H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Z}_\ell(n))_{cotor} := \frac{H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Z}_\ell(n))}{H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Z}_\ell(n))_{tor}}$, and we use that $H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Z}_\ell(n))$ are finitely generated $\mathbb{Z}_\ell$-modules, hence have no nontrivial divisible subgroups.

According to [18, Exposé XXI, 5.5.3], the eigenvalues of the geometric Frobenius acting on $H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Q}_\ell)$ are algebraic integers. After twisting $\mathbb{Q}_\ell$ by n, the eigenvalues will lie in $p^{-n}\overline{\mathbb{Z}}$. Since $n < 0$ by our assumption, this implies that 1 does not appear as an eigenvalue, and hence

$$H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell(n))^{G_{\mathbb{F}_p}} = 0.$$

Thus, after taking the $G_{\mathbb{F}_p}$-invariants in (9), we obtain

$$0 \to (H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div})^{G_{\mathbb{F}_p}} \to H^1(G_{\mathbb{F}_p}, H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))_{cotor}) \to \cdots$$

This gives a monomorphism between the maximal divisible subgroups

$$((H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div})^{G_{\mathbb{F}_p}})_{div} \rightarrowtail H^1(G_{\mathbb{F}_p}, H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))_{cotor})_{div}.$$

However, $H^1(G_{\mathbb{F}_p}, H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))_{cotor})$ is a finitely generated $\mathbb{Z}_\ell$-module, and therefore its maximal divisible subgroup is trivial. We conclude that

$$(H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{F}_p}})_{div} = ((H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div})^{G_{\mathbb{F}_p}})_{div} = 0. \qquad \square$$

Proof of Theorem II. By Definition 1.4, this amounts to showing that the morphism

$$v^*_\infty\colon R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$$

is torsion. The complexes $R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n))$ and $R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$ are almost of cofinite type by Proposition 4.2 and Proposition 3.2 respectively. Therefore, according to Lemma A.4, to show that $v^*_\infty\colon R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))$ is torsion, it suffices to show that the corresponding morphisms on the maximal divisible subgroups

$$H^i_c(v^*_\infty)_{div}\colon H^i_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n))_{div} \to H^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))_{div}$$

are trivial. The morphism $H^i_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n)) \to H^i_c(X_{\overline{\mathbb{Q}},ét}, \mathbb{Q}/\mathbb{Z}(n))$, and hence $H^i_c(v^*_\infty)$, factors through $H^i_c(X_{\overline{\mathbb{Q}},ét}, \mu^{\otimes n})^{G_\mathbb{Q}}$, where $\mu^{\otimes n}$ is the sheaf of all roots of unity on $X_{\overline{\mathbb{Q}},ét}$ twisted by n. So we have

$$\begin{array}{ccc} H^i_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n))_{div} & \xrightarrow{H^i_c(v^*_\infty)_{div}} & H^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(n))_{div} \\ & \searrow \qquad \nearrow & \\ & \left(H^i_c(X_{\overline{\mathbb{Q}},ét}, \mu^{\otimes n})^{G_\mathbb{Q}}\right)_{div} & \end{array}$$

Now

$$\begin{aligned} \left(H^i_c(X_{\overline{\mathbb{Q}},ét}, \mu^{\otimes n})^{G_\mathbb{Q}}\right)_{div} &\cong \left(\bigoplus_\ell H^i_c(X_{\overline{\mathbb{Q}},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_\mathbb{Q}}\right)_{div} \\ &\cong \bigoplus_\ell \left(H^i_c(X_{\overline{\mathbb{Q}},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_\mathbb{Q}}\right)_{div}, \end{aligned}$$

where all the summands are trivial by Proposition 6.2. $\square$

7 Weil-étale complex $R\Gamma_{W,c}(X, \mathbb{Z}(n))$

The aim of this section is to construct the Weil-étale cohomology complexes $R\Gamma_{W,c}(X, \mathbb{Z}(n))$.

LEMMA 7.1. *Let X be an arithmetic scheme and $n < 0$. Assume Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, so that the morphism $\alpha_{X,n}$ exists. Then $u_\infty^* \circ \alpha_{X,n} = 0$.*

$$\begin{array}{ccc}
R\mathrm{Hom}(R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & & \\
\downarrow{\scriptstyle \alpha_{X,n}} & \searrow^{=0} & \\
R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \xrightarrow{u_\infty^*} & R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))
\end{array}$$

PROOF. The morphism $\alpha_{X,n}$ is defined on a complex of $\mathbb{Q}$-vector spaces, and u_∞^* is torsion by Theorem II. □

DEFINITION 7.2. Assuming Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, for $n < 0$, we let $i_\infty^* \colon R\Gamma_{fg}(X, \mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ be a morphism in $\mathbf{D}(\mathbb{Z})$ that gives a morphism of distinguished triangles

$$\begin{array}{ccc}
R\mathrm{Hom}(R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \alpha_{X,n}} & & \downarrow \\
R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \xrightarrow{u_\infty^*} & R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \\
\downarrow & & \downarrow{\scriptstyle id} \\
R\Gamma_{fg}(X, \mathbb{Z}(n)) & \overset{i_\infty^*}{\dashrightarrow} & R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \\
\downarrow & & \downarrow \\
R\mathrm{Hom}(R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) & \longrightarrow & 0
\end{array} \tag{10}$$

In fact, this makes i_∞^* independent of any choices.

PROPOSITION 7.3. *Assuming Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, for $n < 0$, there is a unique morphism i_∞^* that fits in the diagram* (10).

PROOF. We can apply Corollary A.3, since $R\mathrm{Hom}(R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-2])$ is a complex of $\mathbb{Q}$-vector spaces, and both $R\Gamma_{fg}(X, \mathbb{Z}(n))$ and $R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ are almost perfect by Proposition 5.5 and Proposition 3.2. □

PROPOSITION 7.4. *Assuming Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, for $n < 0$, the morphism i_∞^* is torsion.*

Proof. Let us examine the morphism of distinguished triangles (10) that defines i_∞^*; in particular, the commutative diagram

$$\begin{array}{ccc} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X, \mathbb{Z}(n)) \\ {\scriptstyle u_\infty^*}\downarrow & \swarrow {\scriptstyle i_\infty^*} & \\ R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) & & \end{array}$$

According to Corollary A.3, the morphism

$$\begin{aligned} &\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(R\Gamma_{fg}(X, \mathbb{Z}(n)), R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))) \\ &\qquad\qquad \to \operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)), R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))) \end{aligned}$$

induced by the composition with $R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n))$, is a monomorphism, and therefore

$$\begin{aligned} &\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(R\Gamma_{fg}(X, \mathbb{Z}(n)), R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))) \otimes \mathbb{Q} \to \\ &\qquad\qquad \operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)), R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))) \otimes \mathbb{Q} \end{aligned}$$

is also a monomorphism. However, $u_\infty^* \otimes \mathbb{Q} = 0$ by Theorem II, and this implies that $i_\infty^* \otimes \mathbb{Q} = 0$. □

We are now ready to define the Weil-étale complexes.

Definition 7.5. Assuming Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, for $n < 0$, we let $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ be an object in the derived category $\mathbf{D}(\mathbb{Z})$ which is a mapping fiber of i_∞^*:

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \xrightarrow{i_\infty^*} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) \to R\Gamma_{W,c}(X, \mathbb{Z}(n))[1].$$

The **Weil-étale cohomology with compact support** is given by

$$H^i_{W,c}(X, \mathbb{Z}(n)) := H^i(R\Gamma_{W,c}(X, \mathbb{Z}(n))).$$

Remark 7.6. Note that this defines $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ up to a non-unique isomorphism in $\mathbf{D}(\mathbb{Z})$, and the groups $H^i_{W,c}(X, \mathbb{Z}(n))$ are also defined up to a non-unique isomorphism. In a continuation of this paper we will make use of the determinant $\det_\mathbb{Z} R\Gamma_{W,c}(X, \mathbb{Z}(n))$ in the sense of [23], which will be defined up to a canonical isomorphism.

However, we recall from Proposition 5.6 that $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is defined up to a unique isomorphism in the derived category $\mathbf{D}(\mathbb{Z})$. If we could define $i_\infty^* \colon R\Gamma_{fg}(X, \mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n))$ as an explicit, genuine morphism of complexes (not just as a morphism in the derived category $\mathbf{D}(\mathbb{Z})$), this would give us a canonical and functorial definition for $R\Gamma_{W,c}(X, \mathbb{Z}(n))$.

Case of varieties over finite fields

For varieties over finite fields, our Weil-étale cohomology has a simple description, and it is $\mathbb{Q}/\mathbb{Z}$-dual to the arithmetic homology studied by Geisser in [13].

PROPOSITION 7.7. *If X is a variety over a finite field $\mathbb{F}_q$ and $n < 0$, then assuming Conjecture $\mathbf{L}^c(X, n)$, there is an isomorphism of complexes*

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]), \tag{11}$$

and an isomorphism of finite groups

$$\begin{aligned} H^i_{W,c}(X, \mathbb{Z}(n)) &\cong \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \\ &\cong H^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \\ &\cong \mathrm{Hom}(H^c_{i-1}(X_{ar}, \mathbb{Z}(n)), \mathbb{Q}/\mathbb{Z}), \end{aligned}$$

where $H^c_\bullet(X_{ar}, \mathbb{Z}(n))$ are the arithmetic homology groups defined in [13, §3].

PROOF. Under our assumptions, $X(\mathbb{C}) = \emptyset$, and therefore $R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{Z}(n)) = 0$, so that $R\Gamma_{W,c}(X, \mathbb{Z}(n)) \cong R\Gamma_{fg}(X, \mathbb{Z}(n))$. Finally, by Proposition 5.4, we have an isomorphism $R\Gamma_{fg}(X, \mathbb{Z}(n)) \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}[-1])$.

To relate this to Geisser's arithmetic homology, according to [13, Theorem 3.1], there is a long exact sequence

$$\cdots \to H^c_{i-1}(X_{\acute{e}t}, \mathbb{Z}(n)) \to H^c_i(X_{ar}, \mathbb{Z}(n)) \to CH_n(X, i-2n)_\mathbb{Q} \to H^c_{i-2}(X_{\acute{e}t}, \mathbb{Z}(n)) \to \cdots$$

Here the homological notation means that

$$\begin{aligned} H^c_i(X_{\acute{e}t}, \mathbb{Z}(n)) &= H^{-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \\ CH_n(X, i-2n)_\mathbb{Q} &= H^c_i(X_{\acute{e}t}, \mathbb{Q}(n)) = 0, \end{aligned}$$

where the rational vanishing uses finiteness of $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ for X over a finite field and $n < 0$, assuming $\mathbf{L}^c(X_{\acute{e}t}, n)$ (Proposition 4.2).

Therefore,

$$H^c_i(X_{ar}, \mathbb{Z}(n)) \cong H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)).$$

Now (11) gives

$$E_2^{p,q} = \mathrm{Ext}^p_\mathbb{Z}(H^{1-q}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}) \Longrightarrow H^{p+q}_{W,c}(X, \mathbb{Z}(n)),$$

and again, by finiteness of $H^{1-q}(X_{\acute{e}t}, \mathbb{Z}^c(n))$ under our assumptions, this spectral sequence is concentrated in $p = 1$, where

$$\mathrm{Ext}^1_\mathbb{Z}(H^{1-q}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}) \cong \mathrm{Hom}(H^{1-q}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}),$$

so that

$$H^{1+i}_{W,c}(X, \mathbb{Z}(n)) \cong \mathrm{Hom}(H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(H^c_i(X_{ar}, \mathbb{Z}(n)), \mathbb{Q}/\mathbb{Z}). \qquad \square$$

Perfectness of the complex

Our next aim is to verify that $R\Gamma_{W,c}(X,\mathbb{Z}(n))$ is a perfect complex. From now on we tacitly assume Conjecture $\mathbf{L}^c(X_{\acute{e}t},n)$ for $n<0$.

LEMMA 7.8. *The groups $H^i_{W,c}(X,\mathbb{Z}(n))$ are finitely generated for all $i\in\mathbb{Z}$.*

PROOF. In the long exact sequence

$$\cdots\to H^{i-1}_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))\to H^i_{W,c}(X,\mathbb{Z}(n))\to H^i_{fg}(X,\mathbb{Z}(n)) \xrightarrow{H^i(i^*_\infty)} H^i_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))\to\cdots$$

the groups $H^i_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))$ and $H^i_{fg}(X,\mathbb{Z}(n))$ are finitely generated by Proposition 3.2, and Proposition 5.5, respectively. This implies the finite generation of $H^i_{W,c}(X,\mathbb{Z}(n))$. □

LEMMA 7.9. *One has $H^i_{W,c}(X,\mathbb{Z}(n))=0$ for $i<0$.*

PROOF. The definitions of $R\Gamma_{fg}(X,\mathbb{Z}(n))$ and $R\Gamma_{W,c}(X,\mathbb{Z}(n))$ yield exact sequences

$$\begin{array}{ccccccc} & & H^{i-1}_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) & & & & \\ & & \downarrow & & & & \\ & & H^i_{W,c}(X,\mathbb{Z}(n)) & & & & \\ & & \downarrow & & & & \\ H^i_c(X_{\acute{e}t},\mathbb{Z}(n)) & \longrightarrow & H^i_{fg}(X,\mathbb{Z}(n)) & \longrightarrow & \mathrm{Hom}(H^{1-i}(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}) & \to & H^{i+1}_c(X_{\acute{e}t},\mathbb{Z}(n)) \\ & & \downarrow & & & & \\ & & H^i_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)). & & & & \end{array}$$

If $i<0$, then $H^i_c(X_{\acute{e}t},\mathbb{Z}(n))=H^i_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))=0$. Moreover, $\mathrm{Hom}(H^{1-i}(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q})=0$ for $i<0$, since $H^{1-i}(X_{\acute{e}t},\mathbb{Z}^c(n))$ is finite 2-torsion (Proposition 4.2). We conclude that $H^i_{W,c}(X,\mathbb{Z}(n))=H^i_{fg}(X,\mathbb{Z}(n))=0$ for $i<0$. □

For the vanishing of $H^i_{W,c}(X,\mathbb{Z}(n))$ for $i\gg 0$, we first establish the following auxiliary result.

LEMMA 7.10. *Let $d=\dim X$. For each prime ℓ and $i\geq 2d$ we have*

$$H^i_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{Z}_\ell=\widehat{H}^i_c(X[1/\ell]_{\acute{e}t},\mathbb{Z}_\ell(n)), \tag{12}$$

where the right-hand side is defined via $\varprojlim_r \widehat{H}^i_c(X[1/\ell]_{\acute{e}t},\mu_{\ell^r}^{\otimes n})$.

PROOF. Consider the commutative diagram with distinguished rows and columns

$$\begin{array}{ccccccc}
[R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-2]] & \xrightarrow{\widehat{\alpha}_{X,n}} & R\widehat{\Gamma}_c(X_{ét},\mathbb{Z}(n)) & \longrightarrow & R\widehat{\Gamma}_{fg}(X,\mathbb{Z}(n)) & \longrightarrow & [+1] \\
\downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} \\
[R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-2]] & \xrightarrow{\alpha_{X,n}} & R\Gamma_c(X_{ét},\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X,\mathbb{Z}(n)) & \longrightarrow & [+1] \\
\downarrow & & \downarrow{\scriptstyle \widehat{u}_\infty^*} & & \downarrow{\scriptstyle \widehat{i}_\infty^*} & & \downarrow \\
0 & \longrightarrow & R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) & \xrightarrow{id} & R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
[R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-1]] & \xrightarrow{\widehat{\alpha}_{X,n}[1]} & R\widehat{\Gamma}_c(X_{ét},\mathbb{Z}(n))[1] & \longrightarrow & R\widehat{\Gamma}_{fg}(X,\mathbb{Z}(n))[1] & \longrightarrow & [+2]
\end{array}$$

Here $\widehat{u}_\infty^*$ (resp. $\widehat{i}_\infty^*$) is defined as the composition of the canonical morphism u_∞^* (resp. i_∞^*) with the projection to the Tate cohomology

$$\pi\colon R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \to R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)).$$

By Proposition 3.2, $H^i(\pi)$ is an isomorphism for $i \geq 2d-1$. Therefore, the five-lemma applied to

$$\begin{array}{ccccccc}
R\Gamma_{W,c}(X,\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X,\mathbb{Z}(n)) & \xrightarrow{i_\infty^*} & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) & \longrightarrow & [+1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle f[1]} \\
R\widehat{\Gamma}_{fg}(X,\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X,\mathbb{Z}(n)) & \xrightarrow{\widehat{i}_\infty^*} & R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) & \longrightarrow & [+1]
\end{array}$$

shows that for $i \geq 2d$ holds

$$H^i_{W,c}(X,\mathbb{Z}(n)) \cong \widehat{H}^i_{fg}(X,\mathbb{Z}(n)).$$

As in Corollary 5.8, we have for a prime ℓ

$$\widehat{H}^i_{fg}(X,\mathbb{Z}(n)) \otimes \mathbb{Z}_\ell \cong \widehat{H}^i_c(X[1/\ell]_{ét},\mathbb{Z}_\ell(n)).$$ □

COROLLARY 7.11. *One has $H^i_{W,c}(X,\mathbb{Z}(n)) = 0$ for $i > 2d+1$.*

PROOF. It suffices to verify that $H^i_{W,c}(X,\mathbb{Z}(n)) \otimes \mathbb{Z}_\ell = 0$ for each prime ℓ. Thanks to the isomorphism (12), this reduces to $\widehat{H}^i_c(X[1/\ell]_{ét},\mathbb{Z}_\ell(n)) = 0$ for $i > 2d+1$, which is true for reasons of cohomological dimension [1, Exposé X, Théorème 6.2]. We note that if $\ell = 2$ and $X(\mathbb{R}) \neq \emptyset$, then the usual étale cohomology has finite 2-torsion in arbitrarily high degrees. It is important that we consider here the *modified* cohomology with compact support $\widehat{H}^i_c(-)$. To obtain the corresponding statement, combine the arguments from [1, Exposé X] with the well-known computations of modified cohomology for number fields; cf. [30, Chapter II] and [2], [29]. □

Summarizing the above observations, we obtain the following result.

PROPOSITION 7.12. *Conjecture $\mathbf{L}^c(X_{ét}, n)$ implies that $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ for $n < 0$ is a perfect complex. More precisely, $H^i_{W,c}(X, \mathbb{Z}(n))$ are finitely generated groups, and $H^i_{W,c}(X, \mathbb{Z}(n)) = 0$ for $i \notin [0, 2\dim X + 1]$.*

Rational coefficients

PROPOSITION 7.13. *Assuming Conjecture $\mathbf{L}^c(X_{ét}, n)$, for $n < 0$, there is a non-canonical splitting*

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \cong R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q})[-1] \oplus R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Q}(n))[-1].$$

PROOF. The distinguished triangle defining $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ becomes after tensoring with $\mathbb{Q}$

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \xrightarrow{i_\infty^* \otimes \mathbb{Q} = 0} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \otimes \mathbb{Q} \to R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{Q}[1]$$

which yields a non-canonical splitting [37, Chapitre II, Corollaire 1.2.6]

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \cong R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \oplus R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))[-1] \otimes \mathbb{Q},$$

and we have already established in Proposition 5.7 that

$$R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \cong R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q})[-1].$$ □

8 Known cases of Conjecture $\mathbf{L}^c(X_{ét}, n)$

Since the main constructions of this paper assume Conjecture $\mathbf{L}^c(X_{ét}, n)$, we relate it here to other conjectures about the finite generation of étale motivic cohomology formulated in the literature, and also describe certain schemes X for which $\mathbf{L}^c(X_{ét}, n)$ holds unconditionally.

Instead of our $\mathbf{L}^c(X_{ét}, -)$, Flach and Morin state in [8, §3] a slightly different conjecture $\mathbf{L}(X_{ét}, -)$. For proper regular X of pure dimension d, the following is a reformulation of $\mathbf{L}(X_{ét}, d - n)$ [8, Conjecture 3.2, Lemma 3.3] in terms of $\mathbb{Z}^c(n)$.

CONJECTURE 8.1. *For a proper regular arithmetic scheme X and $n < 0$, the groups $H^i(X_{ét}, \mathbb{Z}^c(n))$ are finitely generated for $i \le -2n + 1$.*

A more precise conjectural description of étale motivic cohomology is [15, Conjecture 4.12], which can be written for $\mathbb{Z}^c(n)$ as follows:

Conjecture 8.2. *For a proper regular arithmetic scheme X and $n < 0$, one has*

$$H^i(X_{ét}, \mathbb{Z}^c(n)) = \begin{cases} \textit{finitely generated}, & i \le -2n, \\ \textit{finite}, & i = -2n+1, \\ \textit{cofinite type}, & i \ge -2n+2. \end{cases}$$

This is consistent with our $\mathbf{L}^c(X_{ét}, n)$.

Proposition 8.3. *Let X be a proper regular arithmetic scheme and $n < 0$. Then Conjecture $\mathbf{L}^c(X_{ét}, n)$, Conjecture 8.1, and Conjecture 8.2 are equivalent.*

Proof. For the implication Conjecture 8.1 $\Longrightarrow$ $\mathbf{L}^c(X_{ét}, n)$, by [8, Proposition 3.4], Conjecture 8.1 implies Artin–Verdier duality

$$H^i(X_{ét}, \mathbb{Z}(n)) \cong \operatorname{Hom}(H^{2-i}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \text{ up to finite 2-torsion,}$$

hence $H^i(X_{ét}, \mathbb{Z}^c(n))$ is finite 2-torsion for $i \ge 2$, and in particular for $i > -2n+1$.

The implication Conjecture 8.1 $\Longrightarrow$ Conjecture 8.2 is also established in [8, Proposition 3.4]. □

We now list some special cases where Conjecture $\mathbf{L}^c(X_{ét}, n)$ is known, and therefore gives unconditional results. We follow [32, §5] very closely. For an arithmetic scheme X, we formulate the following conjecture, which is the conjunction of $\mathbf{L}^c(X_{ét}, n)$ for all $n < 0$.

Conjecture 8.4. $\mathbf{L}^c(X_{ét})$: *the cohomology groups $H^i(X_{ét}, \mathbb{Z}^c(n))$ are finitely generated for all $i \in \mathbb{Z}$ and $n < 0$.*

This is similar to [32, Definition 5.8], with the only difference that Morin also requires the finite generation of $H^i(X_{ét}, \mathbb{Z}^c(0))$ for $i \le 0$. Conjecture $\mathbf{L}^c(X_{ét})$ is known for number rings, and also for certain varieties over finite fields. As in [35], [10], and [32], we consider the following class.

Definition 8.5. Let $A(\mathbb{F}_q)$ be the full subcategory of the category of smooth projective varieties over a finite field $\mathbb{F}_q$ generated by products of curves and the following operations.

1) If X and Y lie in $A(\mathbb{F}_q)$, then $X \sqcup Y$ lies $A(\mathbb{F}_q)$.

2) If Y lies in $A(\mathbb{F}_q)$ and there are morphisms $c\colon X \to Y$ and $c'\colon Y \to X$ in the category of Chow motives such that $c' \circ c\colon X \to X$ is a multiplication by constant, then X lies in $A(\mathbb{F}_q)$.

3) If $\mathbb{F}_{q^m}/\mathbb{F}_q$ is a finite extension and $X_{\mathbb{F}_q^m} = X \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}\mathbb{F}_{q^m}$ lies in $A(\mathbb{F}_{q^m})$, then X lies in $A(\mathbb{F}_q)$.

4) If X and Y lie in $A(\mathbb{F}_q)$, and Y is a closed subscheme of X, then the blowup of X along Y lies in $A(\mathbb{F}_q)$.

The following is similar to [32, Definition 5.9].

Definition 8.6. Let $\mathcal{L}(\mathbb{Z})$ be the full subcategory of arithmetic schemes generated by the following objects:

- the empty scheme $\emptyset$,
- $\operatorname{Spec}\mathcal{O}_F$ for a number field F,
- varieties $X \in A(\mathbb{F}_q)$ for any finite field $\mathbb{F}_q$,

and the following operations.

$\mathcal{L}1)$ Let X be an arithmetic scheme, $Z \subset X$ a closed subscheme and $U := X \setminus Z$ its open complement. If two of three schemes X, Z, U lie in $\mathcal{L}(\mathbb{Z})$, then the third also lies in $\mathcal{L}(\mathbb{Z})$.

$\mathcal{L}2)$ A finite disjoint union $X = \coprod_{1\le j\le p} X_j$ lies in $\mathcal{L}(\mathbb{Z})$ if and only if each X_j lies in $\mathcal{L}(\mathbb{Z})$.

$\mathcal{L}3)$ If $V \to U$ is an affine bundle and U lies in $\mathcal{L}(\mathbb{Z})$, then V also lies in $\mathcal{L}(\mathbb{Z})$.

$\mathcal{L}4)$ If $\{U_i \to X\}_{i\in I}$ is a finite surjective family of étale morphisms such that each $U_{i_0,\dots,i_p}$ lies in $\mathcal{L}(\mathbb{Z})$, then X also lies in $\mathcal{L}(\mathbb{Z})$.

Proposition 8.7. *Conjecture $\mathbf{L}^c(X_{\acute{e}t})$ holds for any arithmetic scheme $X \in \mathcal{L}(\mathbb{Z})$.*

Proof. See the argument in [32, Proposition 5.10]. □

Finally, we consider cellular schemes, as in [32, §5.4].

Definition 8.8. Let Y be a separated scheme of finite type over $\operatorname{Spec} k$ for a field k. We say that Y **admits a cellular decomposition** if there exists a filtration of Y by reduced closed subschemes

$$Y^{red} = Y_N \supseteq Y_{N-1} \supseteq \cdots \supseteq Y_{-1} = \emptyset$$

such that $Y_i \setminus Y_{i-1} \cong \mathbb{A}_k^{r_i}$ is isomorphic to an affine space over k.

We say that Y is **geometrically cellular** if $Y_{\overline{k}} = Y \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ admits a cellular decomposition. This is equivalent to the existence of a finite Galois extension k'/k such that $Y_{k'}$ admits a cellular decomposition.

Finally, given an S-scheme $X \to S$ that is separated and of finite type, we say that X is **geometrically cellular over** S if for each $s \in S$ the corresponding fiber X_s is geometrically cellular.

PROPOSITION 8.9. *Let Y be a separated scheme of finite type over $\operatorname{Spec} \mathbb{F}_q$. If Y is geometrically cellular, then $X \in \mathcal{L}(\mathbb{Z})$, and in particular Conjecture $\mathbf{L}^c(Y_{ét})$ holds.*

If $X \to \operatorname{Spec} \mathcal{O}_F$ is a flat, separated scheme of finite type over the ring of integers of a number field, and X is geometrically cellular over $\mathcal{O}_F$, then $X \in \mathcal{L}(\mathbb{Z})$, and in particular $\mathbf{L}^c(X_{ét})$ holds.

For a proof, we refer to [32, Proposition 5.14].

9 Comparison with the complex of Flach and Morin

This paper is based on the ideas of Flach and Morin [8], who gave a similar construction of Weil-étale cohomology $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ for a *proper and regular* arithmetic scheme X, and for *any integer weight* $n \in \mathbb{Z}$. In this section, we will go through the definitions of [8], to verify the following claim.

PROPOSITION 9.1. *Let X be a proper, regular arithmetic scheme, and $n < 0$. Assume Conjecture $\mathbf{L}^c(X_{ét}, n)$. Then the Weil-étale complex $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ defined above in* §7 *is isomorphic to the corresponding complex defined in* [8].

From now on we tacitly assume Conjecture $\mathbf{L}^c(X_{ét}, n)$, which is also equivalent to the assumptions on motivic cohomology in [8] (see Proposition 8.3). Flach and Morin consider the case of a proper and regular arithmetic scheme X of equal dimension d. In this case, we follow [8, Remark 3.11] to reformulate their constructions in terms of complexes $\mathbb{Z}^c(n)$.

Moreover, they work with the Artin–Verdier étale topos $\overline{X}_{ét}$, whose definition and basic properties can be found in [8, §6]. They consider a morphism

$$\overline{\alpha}_{X,n} \colon R\mathrm{Hom}(R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \to R\Gamma(\overline{X}_{ét}, \mathbb{Z}(n)),$$

defined in a similar way to our $\alpha_{X,n}$ (Definition 5.1) using a duality similar to our Theorem I.

The notation in [8] and in this paper is intentionally the same for various objects and morphisms. However, in this section we will write, for example, $\overline{\alpha}_{X,n}$ to denote the morphism of Flach and Morin, to distinguish it from our $\alpha_{X,n}$, etc. An overline indicates that the corresponding thing comes from [8] and has something to do with the Artin–Verdier étale topos.

LEMMA 9.2. *The square*

$$\begin{array}{ccc} R\mathrm{Hom}(R\Gamma(X,\mathbb{Z}^c(n)),\mathbb{Q}[-2]) & \xrightarrow{\overline{\alpha}_{X,n}} & R\Gamma(\overline{X}_{\acute{e}t},\mathbb{Z}(n)) \\ \downarrow{\scriptstyle id} & & \downarrow \\ R\mathrm{Hom}(R\Gamma(X,\mathbb{Z}^c(n)),\mathbb{Q}[-2]) & \xrightarrow{\alpha_{X,n}} & R\Gamma(X_{\acute{e}t},\mathbb{Z}(n)) \end{array} \tag{13}$$

commutes.

PROOF. We recall from Remark 5.2 that $\alpha_{X,n}$ is determined by the maps at the level of cohomology $H^i(\alpha_{X,n})$. The same is true for $\overline{\alpha}_{X,n}$, for the same reasons. Now [8, Theorem 3.5] defines

$$\begin{aligned} H^i(\overline{\alpha}_{X,n})\colon\ & \mathrm{Hom}(H^{2-i}(X,\mathbb{Z}^c(n)),\mathbb{Q}) \xrightarrow{\cong} \mathrm{Hom}(H^{2-i}(\overline{X}_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}) \to \\ & \mathrm{Hom}(H^{2-i}(\overline{X}_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}/\mathbb{Z}) \xleftarrow{\cong} H^i(\overline{X}_{\acute{e}t},\mathbb{Z}(n)), \end{aligned}$$

where the last isomorphism is the duality [8, Corollary 6.26]. Similarly, our morphism $\alpha_{X,n}$ gives

$$\begin{aligned} H^i(\alpha_{X,n})\colon\ & \mathrm{Hom}(H^{2-i}(X,\mathbb{Z}^c(n)),\mathbb{Q}) \xrightarrow{\cong} \mathrm{Hom}(H^{2-i}(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}) \to \\ & \mathrm{Hom}(H^{2-i}(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}/\mathbb{Z}) \xleftarrow{\cong} \widehat{H}^i_c(X_{\acute{e}t},\mathbb{Z}(n)) \to H^i(X_{\acute{e}t},\mathbb{Z}(n)). \end{aligned}$$

The groups $\widehat{H}^i_c(X_{\acute{e}t},\mathbb{Z}(n))$ and $H^i(\overline{X}_{\acute{e}t},\mathbb{Z}(n))$ are different, but the duality in terms of $H^i(\overline{X}_{\acute{e}t},\mathbb{Z}(n))$ is compatible with the duality in terms of $\widehat{H}^i_c(X_{\acute{e}t},\mathbb{Z}(n))$ (see [8, Theorem 6.24]): we have a commutative diagram

$$\begin{array}{ccc} R\widehat{\Gamma}_c(X_{\acute{e}t},\mathbb{Z}/m\mathbb{Z}(n)) & \xrightarrow{\cong} & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}/m\mathbb{Z}^c(n)),\mathbb{Q}/\mathbb{Z}[-2]) \\ \downarrow & & \downarrow \\ R\Gamma(\overline{X}_{\acute{e}t},\mathbb{Z}/m\mathbb{Z}(n)) & \xrightarrow{\cong} & R\mathrm{Hom}(R\Gamma(\overline{X}_{\acute{e}t},\mathbb{Z}/m\mathbb{Z}^c(n)),\mathbb{Q}/\mathbb{Z}[-2]) \end{array}$$

and the diagram

$$\begin{array}{ccc} R\widehat{\Gamma}_c(X_{\acute{e}t},\mathbb{Z}(n)) & \longrightarrow & R\Gamma(X_{\acute{e}t},\mathbb{Z}(n)) \\ \downarrow & \nearrow & \\ R\Gamma(\overline{X}_{\acute{e}t},\mathbb{Z}(n) & & \end{array}$$

commutes as well. We see that the diagram we are interested in commutes:

$$
\begin{array}{ccccccc}
 & & & \xrightarrow{\quad H^i(\overline{\alpha}_{X,n}) \quad} & & & \\
\mathrm{Hom}(H^{2-i}(X,\mathbb{Z}^c(n)),\mathbb{Q}) & \longrightarrow & H^{2-i}(\overline{X}_{\acute{e}t},\mathbb{Z}^c(n))^D & \xleftarrow{\quad\cong\quad} & & & H^i(\overline{X}_{\acute{e}t},\mathbb{Z}(n)) \\
\downarrow{\scriptstyle id} & & \uparrow & & \nearrow & & \downarrow \\
\mathrm{Hom}(H^{2-i}(X,\mathbb{Z}^c(n)),\mathbb{Q}) & \longrightarrow & H^{2-i}(X_{\acute{e}t},\mathbb{Z}^c(n))^D & \xleftarrow{\cong} & \widehat{H}^i_c(X_{\acute{e}t},\mathbb{Z}(n)) & \longrightarrow & H^i(X_{\acute{e}t},\mathbb{Z}(n)) \\
 & & & \xrightarrow{\quad H^i(\alpha_{X,n}) \quad} & & &
\end{array}
$$

For brevity, $\mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z})$ is denoted here by A^D. □

Taking the cones of $\overline{\alpha}_{X,n}$ and $\alpha_{X,n}$, we obtain respectively the complex $R\Gamma_W(\overline{X},\mathbb{Z}(n))$ of Flach and Morin [8, Definition 3.6] and our complex $R\Gamma_{fg}(X,\mathbb{Z}(n))$ (Definition 5.1 above).

The square (13) induces the following diagram with distinguished rows and columns (cf. [33, Proposition 1.4.6]):

(14)

$$
\begin{array}{ccccccc}
[R\Gamma(X,\mathbb{Z}^c(n)),\mathbb{Q}[-2]] & \xrightarrow{\overline{\alpha}_{X,n}} & R\Gamma(\overline{X}_{\acute{e}t},\mathbb{Z}(n)) & \xrightarrow{f} & R\Gamma_W(\overline{X},\mathbb{Z}(n)) & \longrightarrow & [-1] \\
\downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} \\
[R\Gamma(X,\mathbb{Z}^c(n)),\mathbb{Q}[-2]] & \xrightarrow{\alpha_{X,n}} & R\Gamma(X_{\acute{e}t},\mathbb{Z}(n)) & \xrightarrow{g} & R\Gamma_{fg}(X,\mathbb{Z}(n)) & \longrightarrow & [-1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & R\Gamma(X(\mathbb{R}),\tau_{\geq n+1}R\widehat{\pi}_*\mathbb{Z}(n)) & \xrightarrow{id} & R\Gamma(X(\mathbb{R}),\tau_{\geq n+1}R\widehat{\pi}_*\mathbb{Z}(n)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
[R\Gamma(X,\mathbb{Z}^c(n)),\mathbb{Q}[-1]] & \longrightarrow & R\Gamma(\overline{X}_{\acute{e}t},\mathbb{Z}(n))[1] & \xrightarrow{f[1]} & R\Gamma_W(\overline{X},\mathbb{Z}(n))[1] & \longrightarrow & [0]
\end{array}
$$

Then [8, Definition 3.23] considers a morphism $\overline{u}^*_\infty$ defined via

(15)

$$
\begin{array}{ccccccc}
R\Gamma(\overline{X}_{\acute{e}t},\mathbb{Z}(n)) & \longrightarrow & R\Gamma(X_{\acute{e}t},\mathbb{Z}(n)) & \longrightarrow & R\Gamma(X(\mathbb{R}),\tau_{\geq n+1}R\widehat{\pi}_*\mathbb{Z}(n)) & \to & [+1] \\
\exists\downarrow{\scriptstyle \overline{u}^*_\infty} & & \downarrow{\scriptstyle u^*_\infty} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \overline{u}^*_\infty[1]} \\
R\Gamma_W(X_\infty,\mathbb{Z}(n)) & \to & R\Gamma(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) & \to & R\Gamma(X(\mathbb{R}),\tau_{\geq n+1}R\widehat{\pi}_*\mathbb{Z}(n)) & \to & [+1]
\end{array}
$$

Here the complex $R\Gamma_W(X_\infty,\mathbb{Z}(n))$ is *defined* via the bottom triangle.

Then [8, Proposition 3.24] and our Proposition 7.3 above establish the existence and uniqueness of morphisms $\overline{\iota}^*_\infty$ and i^*_∞ which make the triangles below commutative, and then the Weil-étale complexes are defined as mapping fibers of $\overline{\iota}^*_\infty$ and i^*_∞:

$$
\begin{array}{ccc}
R\Gamma_{W,c}(\overline{X},\mathbb{Z}(n)) & & \\
\downarrow & & \\
R\Gamma_W(\overline{X},\mathbb{Z}(n)) & \xleftarrow{f} & R\Gamma(\overline{X}_{ét},\mathbb{Z}(n)) \\
\Big\downarrow{\scriptstyle \overline{\iota}_\infty^*} & \swarrow{\scriptstyle \overline{u}_\infty^*} & \\
R\Gamma_W(X_\infty,\mathbb{Z}(n)) & & \\
\downarrow & & \\
R\Gamma_{W,c}(\overline{X},\mathbb{Z}(n))[1] & &
\end{array}
\qquad
\begin{array}{ccc}
R\Gamma_{W,c}(X,\mathbb{Z}(n)) & & \\
\downarrow & & \\
R\Gamma_{fg}(X,\mathbb{Z}(n)) & \xleftarrow{g} & R\Gamma(X_{ét},\mathbb{Z}(n)) \\
\Big\downarrow{\scriptstyle \iota_\infty^*} & \swarrow{\scriptstyle u_\infty^*} & \\
R\Gamma(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) & & \\
\downarrow & & \\
R\Gamma_{W,c}(X,\mathbb{Z}(n))[1] & &
\end{array}
$$

In order to compare the two resulting complexes, we note that $\overline{u}_\infty^*$ is only defined via (15), so in the diagram below from Figure 1, we can first choose $\overline{\iota}_\infty^*$ such that the front face gives a morphism of triangles. Then we can *declare* $\overline{u}_\infty^*$ to be the composition $\overline{\iota}_\infty^* \circ f$. In this way everything commutes, and we see that $R\Gamma_{W,c}(\overline{X},\mathbb{Z}(n)) \cong R\Gamma_{W,c}(X,\mathbb{Z}(n))$.

This concludes the proof of Proposition 9.1. $\square$

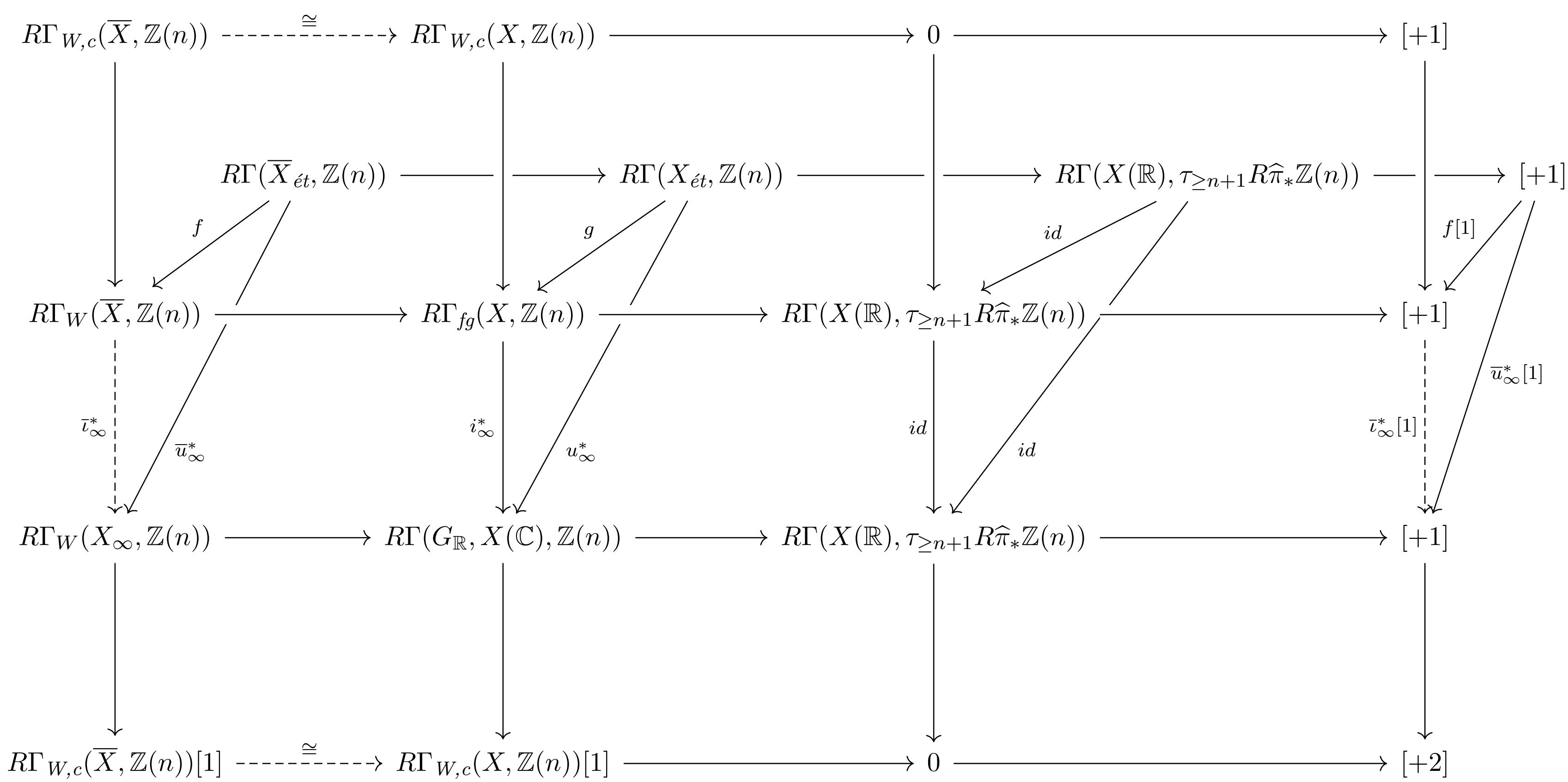

Figure 1: Comparison of the Weil-étale complexes from [8] and this paper, denoted $R\Gamma_{W,c}(\overline{X}, \mathbb{Z}(n))$ and $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ respectively. The top face of the prism comes from (14). The arrow $\overline{\iota}_\infty^*$ is chosen so that the front face is commutative. Then set $\overline{u}_\infty^* = \overline{\iota}_\infty^* \circ f$ so that the back face is commutative and corresponds to (15).

A Some homological algebra

This appendix contains some basic results about the derived category of abelian groups $\mathbf{D}(\mathbb{Z})$ which are used throughout the text. The following lemmas are isolated from the proofs in [8], with some modifications to treat the 2-torsion.

First, recall that every complex of abelian groups $A^\bullet$ (not necessarily bounded) is quasi-isomorphic to its cohomology:

$$A^\bullet \cong \coprod_{i\in\mathbb{Z}} H^i(A^\bullet)[-i] \cong \prod_{i\in\mathbb{Z}} H^i(A^\bullet)[-i] \\ = \Big(\cdots \to H^{i-1}(A^\bullet) \xrightarrow{0} H^i(A^\bullet) \xrightarrow{0} H^{i+1}(A^\bullet) \to \cdots\Big).$$

Here $\coprod_{i\in\mathbb{Z}} H^i(A^\bullet)[-i] \cong \prod_{i\in\mathbb{Z}} H^i(A^\bullet)[-i]$ is the complex that has $H^i(A^\bullet)$ in i-th degree, which serves as both product and coproduct of complexes $H^i(A^\bullet)[-i]$ concentrated in i-th degree. This gives us a useful expression for morphisms in the derived category: since $\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(A, B[i]) \cong \operatorname{Ext}^i_{\mathbb{Z}}(A,B)$, and $\operatorname{Ext}^i_{\mathbb{Z}}(A,B) = 0$ for $i>1$, we obtain

$$\begin{aligned} \operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, B^\bullet) &\cong \operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(\coprod_{i\in\mathbb{Z}} H^i(A^\bullet)[-i], \prod_{j\in\mathbb{Z}} H^j(B^\bullet)[-j]) \\ &\cong \prod_{i\in\mathbb{Z}}\prod_{j\in\mathbb{Z}} \operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(H^i(A^\bullet), H^j(B^\bullet)[i-j]) \\ &\cong \prod_{i\in\mathbb{Z}} \big(\operatorname{Hom}(H^i(A^\bullet), H^i(B^\bullet)) \oplus \operatorname{Ext}(H^i(A^\bullet), H^{i-1}(B^\bullet))\big) \\ &\cong \prod_{i\in\mathbb{Z}} \operatorname{Hom}(H^i(A^\bullet), H^i(B^\bullet)) \oplus \prod_{i\in\mathbb{Z}} \operatorname{Ext}(H^i(A^\bullet), H^{i-1}(B^\bullet)). \end{aligned} \tag{16}$$

LEMMA A.1.

1) *If $C^\bullet$ and $C'^\bullet$ are almost perfect in the sense of Definition* 1.1*, then the group $\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(C^\bullet, C'^\bullet)$ has no nontrivial divisible subgroups.*

2) *If $A^\bullet$ is a complex such that $H^i(A^\bullet)$ are finite-dimensional $\mathbb{Q}$-vector spaces and $C^\bullet$ is a complex such that $H^i(C^\bullet)$ are finitely generated abelian groups, then the group $\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, C^\bullet)$ is divisible.*

PROOF. In 1), if $C^\bullet$ and $C'^\bullet$ are almost perfect, then $\operatorname{Hom}(H^i(C^\bullet), H^i(C'^\bullet))$ are finitely generated groups, 2-torsion for $i \gg 0$. Writing $H^i(C^\bullet) \cong \mathbb{Z}^{\oplus r} \oplus G$, $H^{i-1}(C'^\bullet) \cong \mathbb{Z}^{\oplus r'} \oplus G'$ for some r, r' and finite groups G, G', we calculate that

$$\operatorname{Ext}(\mathbb{Z}^{\oplus r} \oplus G, \mathbb{Z}^{\oplus r'} \oplus G') \cong \underbrace{\operatorname{Ext}(G,\mathbb{Z})}_{\cong G}{}^{\oplus r'} \oplus \operatorname{Ext}(G, G')$$

are finite groups, 2-torsion for $i \gg 0$. It follows from (16) that $\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(C^\bullet, C'^\bullet)$ is a sum of a finitely generated group and a 2-torsion group, so it cannot have nontrivial divisible subgroups.

Similarly in 2), under our assumption, $\operatorname{Hom}(H^i(A^\bullet), H^i(C^\bullet)) = 0$ for all i, and the calculation

$$\operatorname{Ext}(\mathbb{Q}^{\oplus r}, \mathbb{Z}^{\oplus s} \oplus G) \cong \operatorname{Ext}(\mathbb{Q}, \mathbb{Z})^{\oplus rs} \oplus \underbrace{\operatorname{Ext}(\mathbb{Q}, G)}_{=0}{}^{\oplus r}$$

shows that $\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, C^\bullet)$ is a direct product of divisible groups $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z})$, hence divisible. □

Recall that Verdier's axiom (TR1) states that every morphism $v\colon A^\bullet \to B^\bullet$ can be completed to a distinguished triangle $A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$. Axiom (TR3) states that for every commutative diagram with distinguished rows

(17)
$$\begin{array}{ccccccc}
A^\bullet & \xrightarrow{u} & B^\bullet & \xrightarrow{v} & C^\bullet & \xrightarrow{w} & A^\bullet[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \\
A'^\bullet & \xrightarrow{u'} & B'^\bullet & \xrightarrow{v'} & C'^\bullet & \xrightarrow{w'} & A'^\bullet[1]
\end{array}$$

there exists some $h\colon C^\bullet \to C'^\bullet$, which gives a morphism of distinguished triangles

(18)
$$\begin{array}{ccccccc}
A^\bullet & \xrightarrow{u} & B^\bullet & \xrightarrow{v} & C^\bullet & \xrightarrow{w} & A^\bullet[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \exists h} & & \downarrow{\scriptstyle f[1]} \\
A'^\bullet & \xrightarrow{u'} & B'^\bullet & \xrightarrow{v'} & C'^\bullet & \xrightarrow{w'} & A'^\bullet[1]
\end{array}$$

The cone $C^\bullet$ in (TR1) and the morphism h in (TR3) are neither unique nor canonical. Two different cones of the same morphism are necessarily isomorphic, but the isomorphism between them is not unique, because it is provided by (TR3). Let us recall a useful argument showing that things are well-defined in some special cases.

Lemma A.2 (≈[3, Proposition 1.1.9, Corollaire 1.1.10]). *Consider the derived category* $\mathbf{D}(\mathcal{A})$ *of an abelian category* $\mathcal{A}$.

1) *For a commutative diagram* (17), *assume that the homomorphism of abelian groups*

$$w^*\colon \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet[1], C'^\bullet) \to \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(C^\bullet, C'^\bullet)$$

induced by w is trivial. Then there exists a unique morphism $h\colon C^\bullet \to C'^\bullet$ that gives a morphism of triangles (18).

2) *For a distinguished triangle $A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$, assume that for any other cone $C'^\bullet$ of u the morphism w^* is trivial. Then the cone of u is unique up to a unique isomorphism.*

Proof. In 1), applying $\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(-, C'^\bullet)$ to the first distinguished triangle, we obtain an exact sequence of abelian groups

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet[1], C'^\bullet) \xrightarrow{w^*} \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(C^\bullet, C'^\bullet) \xrightarrow{v^*} \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(B^\bullet, C'^\bullet).$$

If $w^* = 0$, we conclude that v^* is a monomorphism. This implies that there is a unique morphism h such that $h \circ v = v' \circ g$. Now in 2), if $C^\bullet$ and $C'^\bullet$ are two different cones of u, we have a commutative diagram

$$\begin{array}{ccccccc}
A^\bullet & \xrightarrow{u} & B^\bullet & \xrightarrow{v} & C^\bullet & \xrightarrow{w} & A^\bullet[1] \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \vdots & & \downarrow{\scriptstyle id} \\
A^\bullet & \xrightarrow{u'} & B^\bullet & \xrightarrow{v'} & C'^\bullet & \xrightarrow{w'} & A^\bullet[1]
\end{array}$$

By the triangulated five-lemma, the dashed arrow is an isomorphism, and it is unique thanks to part 1). □

Here is a special case that we need.

Corollary A.3. *Consider the derived category $\mathbf{D}(\mathbb{Z})$.*

1) *Suppose we have a commutative diagram with distinguished rows* (17)*, where $A^\bullet$ is a complex such that $H^i(A^\bullet)$ are finite-dimensional $\mathbb{Q}$-vector spaces and $C^\bullet$, $C'^\bullet$ are almost perfect complexes in the sense of Definition* 1.1*. Then there exists a unique morphism $h\colon C^\bullet \to C'^\bullet$ which gives a morphism of triangles* (18).

2) *For a distinguished triangle*

$$A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$$

assume that $A^\bullet$ is a complex such that $H^i(A^\bullet)$ are finite-dimensional $\mathbb{Q}$-vector spaces and $C^\bullet$ is an almost perfect complex. Then the cone of u is unique up to a unique isomorphism.

Proof. In this situation, by Lemma A.1, the group $\mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(C^\bullet, C'^\bullet)$ has no nontrivial divisible subgroups, and $\mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet[1], C'^\bullet)$ is divisible. This means that there are no nontrivial homomorphisms $\mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet[1], C'^\bullet) \to \mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(C^\bullet, C'^\bullet)$, and we can apply Lemma A.2. □

Lemma A.4. *Suppose that $A^\bullet$ and $B^\bullet$ are almost of cofinite type in the sense of Definition* 1.1*. Then a morphism $f\colon A^\bullet \to B^\bullet$ is torsion (i.e. a torsion element in the group $\mathrm{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, B^\bullet)$, i.e. $f \otimes \mathbb{Q} = 0$) if and only if the morphisms $H^i(f)\colon H^i(A^\bullet) \to H^i(B^\bullet)$ are torsion; that is, they are trivial on the maximal divisible subgroups:*

$$(H^i(f)_{div}\colon H^i(A^\bullet)_{div} \to H^i(B^\bullet)_{div}) = 0.$$

PROOF. We may write $H^i(A^\bullet) \cong (\mathbb{Q}/\mathbb{Z})^{\oplus r} \oplus G$ and $H^{i-1}(B^\bullet) \cong (\mathbb{Q}/\mathbb{Z})^{\oplus s} \oplus H$ for some r, s and some finite groups G, H. Now

$$\operatorname{Ext}((\mathbb{Q}/\mathbb{Z})^{\oplus r} \oplus G, (\mathbb{Q}/\mathbb{Z})^{\oplus s} \oplus H) \cong \underbrace{\operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, H)}_{\cong H}{}^{\oplus r} \oplus \operatorname{Ext}(G, H)$$

is a finite group. It follows that tensoring (16) with $\mathbb{Q}$ kills $\prod_{i\in\mathbb{Z}} \operatorname{Ext}(H^i(A^\bullet), H^{i-1}(B^\bullet))$ and gives an isomorphism

$$\begin{gathered}\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, B^\bullet) \otimes \mathbb{Q} \cong \prod_{i\in\mathbb{Z}} \operatorname{Hom}(H^i(A^\bullet), H^i(B^\bullet)) \otimes \mathbb{Q}, \\ f \otimes \mathbb{Q} \mapsto (H^i(f) \otimes \mathbb{Q})_{i\in\mathbb{Z}}.\end{gathered}$$

□

LEMMA A.5. *If $A^\bullet$ is a complex of $\mathbb{Q}$-vector spaces and $B^\bullet$ is a complex almost of cofinite type in the sense of Definition* 1.1*, then there is an isomorphism of abelian groups*

$$\begin{gathered}\operatorname{Hom}_{\mathbf{D}(\mathbb{Z})}(A^\bullet, B^\bullet) \xrightarrow{\cong} \prod_{i\in\mathbb{Z}} \operatorname{Hom}(H^i(A^\bullet), H^i(B^\bullet)), \\ f \mapsto (H^i(f))_{i\in\mathbb{Z}}.\end{gathered}$$

PROOF. If $H^i(A^\bullet)$ are $\mathbb{Q}$-vector spaces and $H^{i-1}(B^\bullet)$ are groups of cofinite type, then the term $\operatorname{Ext}(H^i(A^\bullet), H^{i-1}(B^\bullet))$ in the formula (16) vanishes by calculations similar to the above, as $\operatorname{Ext}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) = \operatorname{Ext}(\mathbb{Q}, G) = 0$ for finite G. □

B Cohomology with compact support

For any arithmetic scheme $f\colon X \to \operatorname{Spec}\mathbb{Z}$ there exists a **Nagata compactification** [6, 7] (see also [1, Exposé XVII])

$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & \mathfrak{X} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & \operatorname{Spec}\mathbb{Z} & \end{array}$$

where j is an open immersion and g is a proper morphism.

DEFINITION B.1. Let X be an arithmetic scheme and let $\mathcal{F}$ be an abelian torsion sheaf on $X_{\acute{e}t}$. Then one defines the **cohomology with compact support** of $\mathcal{F}$ via the complex

$$R\Gamma_c(X_{\acute{e}t}, \mathcal{F}) := R\Gamma(\mathfrak{X}_{\acute{e}t}, j_!\mathcal{F}).$$

For torsion sheaves, this does not depend on the choice of $j\colon X \hookrightarrow \mathfrak{X}$, but here we would like to fix this choice in order to compare cohomology with compact support on $X_{\acute{e}t}$ with the singular cohomology with compact support on $X(\mathbb{C})$.

Comparison with the analytic cohomology

DEFINITION B.2. Given a Nagata compactification $j\colon X \hookrightarrow \mathfrak{X}$, we consider the corresponding open immersion $j(\mathbb{C})\colon X(\mathbb{C}) \to \mathfrak{X}(\mathbb{C})$, and for a sheaf $\mathcal{F}$ on $X(\mathbb{C})$ we define

$$\Gamma_c(X(\mathbb{C}), \mathcal{F}) := \Gamma(\mathfrak{X}(\mathbb{C}), j(\mathbb{C})_! \mathcal{F}).$$

Similarly, for a $G_\mathbb{R}$-equivariant sheaf on $X(\mathbb{C})$ we define

$$\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathcal{F}) := \Gamma(G_\mathbb{R}, \mathfrak{X}(\mathbb{C}), j(\mathbb{C})_! \mathcal{F}).$$

The canonical reference for the comparison between étale and singular cohomology is [1, Exposé XI, §4], so we borrow some definitions and notations from there. Let X be an arithmetic scheme.

1. The base change from $\operatorname{Spec} \mathbb{Z}$ to $\operatorname{Spec} \mathbb{C}$ gives us a morphism of sites

$$\gamma\colon X_{\mathbb{C},ét} \to X_{ét}.$$

2. Let X_{cl} be the site of étale maps $f\colon U \to X(\mathbb{C})$. A covering family in X_{cl} is a family of maps $\{U_i \to U\}$ such that U is the union of images of U_i.

   (We recall that in the analytic topology, $f\colon U \to X(\mathbb{C})$ is **étale** if it is a *local on the source homeomorphism*: for each $u \in U$ there exists an open neighborhood $u \ni V$ such that $f|_V : V \to f(V)$ is a homeomorphism.)

   Since the inclusion of an open subset $U \subset X(\mathbb{C})$ is an étale map, we have a fully faithful functor $X(\mathbb{C}) \subset X_{cl}$, and the topology on $X(\mathbb{C})$ is induced by the topology on X_{cl}. This gives us a morphism of sites $\delta\colon X_{cl} \to X(\mathbb{C})$, which by the comparison lemma [1, Exposé III, Théorème 4.1] induces an equivalence of the corresponding categories of sheaves

$$\delta_*\colon \mathbf{Sh}(X_{cl}) \to \mathbf{Sh}(X(\mathbb{C})).$$

3. A morphism of schemes $f\colon X'_\mathbb{C} \to X_\mathbb{C}$ over $\operatorname{Spec} \mathbb{C}$ is étale if and only if the map $f(\mathbb{C})\colon X'(\mathbb{C}) \to X(\mathbb{C})$ is étale [19, Exposé XII, Proposition 3.1], and therefore the functor $X'_\mathbb{C} \rightsquigarrow X'(\mathbb{C})$ gives us a morphism of sites

$$\epsilon\colon X_{cl} \to X_{\mathbb{C},ét}.$$

DEFINITION B.3. We define the functor

$$\alpha^*\colon \mathbf{Sh}(X_{ét}) \to \mathbf{Sh}(G_\mathbb{R}, X(\mathbb{C}))$$

via the composition

$$\mathbf{Sh}(X_{ét}) \xrightarrow{\gamma^*} \mathbf{Sh}(X_{\mathbb{C},ét}) \xrightarrow{\epsilon^*} \mathbf{Sh}(X_{cl}) \xrightarrow[\simeq]{\delta_*} \mathbf{Sh}(X(\mathbb{C}))$$

As we start from a scheme over $\operatorname{Spec}\mathbb{Z}$ and base change to $\operatorname{Spec}\mathbb{C}$, the resulting sheaf on $X(\mathbb{C})$ is equivariant with respect to the complex conjugation, hence an object in $\mathbf{Sh}(G_\mathbb{R}, X(\mathbb{C}))$. For the definition of equivariant sheaves, we refer to the introduction.

LEMMA B.4. *α^* preserves colimits.*

PROOF. α^* is the composition of the inverse image functors γ^* and ϵ^* (which are left adjoint) and an equivalence δ_*. □

PROPOSITION B.5. *Given a sheaf $\mathcal{F}$ on $X_{ét}$, there exists a natural morphism*

$$\Gamma(X_{ét}, \mathcal{F}) \to \Gamma(G_\mathbb{R}, X(\mathbb{C}), \alpha^*\mathcal{F}),$$

and similarly, for cohomology with compact support,

$$\Gamma_c(X_{ét}, \mathcal{F}) \to \Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \alpha^*\mathcal{F}).$$

PROOF. If $j\colon X \hookrightarrow \mathfrak{X}$ is a Nagata compactification, we have the corresponding compactification $j(\mathbb{C})\colon X(\mathbb{C}) \hookrightarrow \mathfrak{X}(\mathbb{C})$. The extension by zero morphism $j(\mathbb{C})_!\colon \mathbf{Sh}(X(\mathbb{C})) \to \mathbf{Sh}(\mathfrak{X}(\mathbb{C}))$ restricts to the subcategory of $G_\mathbb{R}$-equivariant sheaves: if $\mathcal{F}$ is a $G_\mathbb{R}$-equivariant sheaf on $X(\mathbb{C})$, then $j(\mathbb{C})_!\mathcal{F}$ is a $G_\mathbb{R}$-equivariant sheaf on $\mathfrak{X}(\mathbb{C})$. From the definition of α^*, we see that that there is a commutative diagram

$$\begin{array}{ccc}
\mathbf{Sh}(X_{ét}) & \xrightarrow{\alpha^*} & \mathbf{Sh}(G_\mathbb{R}, X(\mathbb{C})) \\
{\scriptstyle j_!}\downarrow & & \downarrow{\scriptstyle j(\mathbb{C})_!} \\
\mathbf{Sh}(\mathfrak{X}_{ét}) & \xrightarrow[\alpha^*_\mathfrak{X}]{} & \mathbf{Sh}(G_\mathbb{R}, \mathfrak{X}(\mathbb{C}))
\end{array}$$

—this diagram commutes for representable étale sheaves, and then every étale sheaf is a colimit of representable sheaves, and α^*, $j_!$, $\alpha^*_\mathfrak{X}$, $j(\mathbb{C})_!$ preserve colimits, as left adjoints.

The morphism in question is given by

$$\begin{aligned}
\Gamma_c(X_{ét}, \mathcal{F}) := \Gamma(\mathfrak{X}_{ét}, j_!\mathcal{F}) &\to \Gamma(G_\mathbb{R}, \mathfrak{X}(\mathbb{C}), \alpha^*_\mathfrak{X} j_!\mathcal{F}) \\
&= \Gamma(G_\mathbb{R}, \mathfrak{X}(\mathbb{C}), j(\mathbb{C})_!\, \alpha^*\mathcal{F}) =: \Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \alpha^*\mathcal{F}). \quad \square
\end{aligned}$$

The morphism α is also discussed in [8, Appendix A], but Flach and Morin work with proper schemes; the above remarks are to make sure that everything works fine for compactifications.

Modified étale cohomology

Here we briefly review the **modified étale cohomology with compact support** $R\widehat{\Gamma}_c(X_{\acute{e}t}, -)$. It was introduced by Th. Zink in [20, Appendix 2] for the case of number rings $X = \operatorname{Spec} \mathcal{O}_{K,S}$, and it is also discussed in [30, §II.2]. The general definition for $X \to \operatorname{Spec} \mathbb{Z}$ is treated in [8, §6.7] and [16, §2].

Thanks to the Leray spectral sequence $R\Gamma(\mathfrak{X}_{\acute{e}t}, -) \cong R\Gamma(\operatorname{Spec} \mathbb{Z}_{\acute{e}t}, -) \circ Rg_*$, we have

$$R\Gamma_c(X_{\acute{e}t}, \mathcal{F}) := R\Gamma(\mathfrak{X}_{\acute{e}t}, j_! \mathcal{F}) \cong R\Gamma((\operatorname{Spec} \mathbb{Z})_{\acute{e}t}, Rf_! \mathcal{F}), \quad \text{where } Rf_! \mathcal{F} := Rg_* j_! \mathcal{F}.$$

First we recall that for a finite group G and a G-module A the corresponding group cohomology $H^i(G, A)$ (resp. Tate cohomology $\widehat{H}^i(G, A)$) can be defined in terms of resolutions $P_\bullet$ (resp. complete resolutions $\widehat{P}_\bullet$) of $\mathbb{Z}$ by free $\mathbb{Z}G$-modules (see e.g. [5, Chapter VI]). More generally, if $A^\bullet$ is a bounded (cohomological) complex of G-modules, we obtain a *double complex* of abelian groups $\operatorname{Hom}^{\bullet\bullet}(P_\bullet, A^\bullet)$ (resp. $\operatorname{Hom}^{\bullet\bullet}(\widehat{P}_\bullet, A^\bullet)$), and it makes sense to define the corresponding **group hypercohomology** (resp. **Tate hypercohomology**) via the complexes

$$R\Gamma(G, A^\bullet) := \operatorname{Tot}^\oplus(\operatorname{Hom}^{\bullet\bullet}(P_\bullet, A^\bullet)), \quad R\widehat{\Gamma}(G, A^\bullet) := \operatorname{Tot}^\oplus(\operatorname{Hom}^{\bullet\bullet}(\widehat{P}_\bullet, A^\bullet)).$$

Now if $\mathcal{F}$ is an abelian sheaf on $(\operatorname{Spec} \mathbb{Z})_{\acute{e}t}$, then the corresponding **modified cohomology with compact support** is characterized by the distinguished triangle

$$R\widehat{\Gamma}_c((\operatorname{Spec} \mathbb{Z})_{\acute{e}t}, \mathcal{F}) \to R\Gamma((\operatorname{Spec} \mathbb{Z})_{\acute{e}t}, \mathcal{F}) \to R\widehat{\Gamma}(G_\mathbb{R}, v^* \mathcal{F}) \to R\widehat{\Gamma}_c((\operatorname{Spec} \mathbb{Z})_{\acute{e}t}, \mathcal{F})[1].$$

Here $v\colon \operatorname{Spec} \mathbb{R} \to \operatorname{Spec} \mathbb{Z}$ is the canonical morphism, and $v^* \mathcal{F}$ is the corresponding sheaf on $(\operatorname{Spec} \mathbb{R})_{\acute{e}t}$, which can be viewed as a $G_\mathbb{R}$-module by [1, Exposé VII, 2.3], and $R\widehat{\Gamma}(G_\mathbb{R}, v^* \mathcal{F})$ denotes the corresponding Tate cohomology.

In general, given an arithmetic scheme $X \to \operatorname{Spec} \mathbb{Z}$ and a torsion abelian sheaf $\mathcal{F}$ on $X_{\acute{e}t}$, we choose a Nagata compactification as above and set

$$R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathcal{F}) := R\widehat{\Gamma}_c((\operatorname{Spec} \mathbb{Z})_{\acute{e}t}, Rf_! \mathcal{F}).$$

We have a natural morphism

$$R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathcal{F}) \to R\Gamma_c(X_{\acute{e}t}, \mathcal{F}),$$

which is an isomorphism if $X(\mathbb{R}) = \emptyset$. In general, Tate cohomology $\widehat{H}^i(G_\mathbb{R}, -)$ is annihilated by multiplication by $2 = \# G_\mathbb{R}$, and therefore $\widehat{H}^i_c(X_{\acute{e}t}, \mathcal{F}) \to H^i_c(X_{\acute{e}t}, \mathcal{F})$ has 2-torsion kernel and cokernel.

For canonicity and functoriality, I refer to [16, §2].

References

[1] Michael Artin, Alexander Grothendieck and Jean-Louis Verdier (eds.), Séminaire de géométrie algébrique du Bois-Marie 1963–1964 (SGA 4): Théorie des topos et cohomologie étale des schémas, Lecture Notes in Mathematics, Vol. 269, 270, 305, Springer-Verlag, Berlin-New York, 1972–73, Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[2] Michael Artin and Jean-Louis Verdier, Seminar on étale cohomology of number fields, Lecture notes prepared in connection with the seminars held at the summer institute on algebraic geometry. Whitney estate, Woods Hole, Massachusetts. July 6 – July 31, 1964, Amer. Math. Soc., Providence, RI, 1964.

[3] A. A. Beilinson, J. Bernstein and P. Deligne, Faisceaux pervers, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171.

[4] Spencer Bloch, Algebraic cycles and higher K-theory, Adv. in Math. **61** (1986), 267–304

[5] Kenneth S. Brown, Cohomology of groups, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original.

[6] Brian Conrad, Deligne's notes on Nagata compactifications, J. Ramanujan Math. Soc. **22** (2007), no. 3, 205–257.

[7] Brian Conrad, Erratum for "Deligne's notes on Nagata compactifications", J. Ramanujan Math. Soc. **24** (2009), no. 4, 427–428.

[8] Matthias Flach and Baptiste Morin, Weil-étale cohomology and zeta-values of proper regular arithmetic schemes, Doc. Math. **23** (2018), 1425–1560.

[9] Thomas Geisser, Motivic cohomology over Dedekind rings, Math. Z. **248** (2004), no. 4, 773–794.

[10] Thomas Geisser, Weil-étale cohomology over finite fields, Math. Ann. **330** (2004), no. 4, 665–692.

[11] Thomas Geisser, Motivic cohomology, K-theory and topological cyclic homology, in *Handbook of K-Theory*. Vol. 1, 2, Springer-Verlag, New York, 2005, 193–234.

[12] Thomas Geisser, Arithmetic cohomology over finite fields and special values of ζ-functions, Duke Math. J. **133** (2006), no. 1, 27–57.

[13] Thomas Geisser, Arithmetic homology and an integral version of Kato's conjecture, J. Reine Angew. Math. **644** (2010), 1–22.

[14] Thomas Geisser, Duality via cycle complexes, Ann. of Math. (2) **172** (2010), no. 2, 1095–1126.

[15] Thomas Geisser, On the structure of étale motivic cohomology, J. Pure Appl. Algebra **221** (2017), no. 7, 1614–1628.

[16] Thomas Geisser and Alexander Schmidt, Poitou-Tate duality for arithmetic schemes, Compos. Math. **154** (2018), no. 9, 2020–2044.

[17] Alexander Grothendieck (ed.), Séminaire de géométrie algébrique du Bois-Marie 1965–66 (SGA 5): Cohomologie ℓ-adique et fonctions L, Lecture Notes in Mathematics, Vol. 589, Springer-Verlag, Berlin-New York, 1977, Avec la collaboration de I. Bucur, C. Houzel, L. Illusie, J.-P. Jouanolou et J.-P. Serre.

[18] Alexander Grothendieck, Pierre Deligne and Nicholas Katz (eds.), Séminaire de géométrie algébrique du Bois-Marie 1967–69 (SGA 7): Groupes de monodromie en géométrie algébrique, Lecture Notes in Mathematics, Vol. 288, 340, Springer-Verlag, Berlin-New York, 1972–73, Avec la collaboration de M. Raynaud et D.S. Rim.

[19] Alexander Grothendieck and Michèle Raynaud (eds.), Séminaire de géométrie algébrique du Bois-Marie 1960–61 (SGA 4): Revêtements étales et groupe fondamental, Lecture Notes in Mathematics, Vol. 224, Springer-Verlag, Berlin-New York, 1971.

[20] Klaus Haberland, Galois cohomology of algebraic number fields, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978, With two appendices by Helmut Koch and Thomas Zink.

[21] Heisuke Hironaka, Triangulations of algebraic sets. Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), pp. 165–185. Amer. Math. Soc., Providence, R.I., 1975.

[22] Bruno Kahn, Some finiteness results for étale cohomology, J. Number Theory **99** (2003), no. 1, 57–73.

[23] Finn Faye Knudsen and David Mumford, The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div", Math. Scand. **39** (1976), no. 1, 19–55.

[24] Stephen Lichtenbaum, The Weil-étale topology on schemes over finite fields, Compos. Math. **141** (2005), no. 3, 689–702.

[25] Stephen Lichtenbaum, Euler characteristics and special values of zeta-functions, Motives and algebraic cycles, Fields Inst. Commun., vol. 56, Amer. Math. Soc., Providence, RI, 2009, pp. 249–255.

[26] Stephen Lichtenbaum, The Weil-étale topology for number rings, Ann. of Math. (2) **170** (2009), no. 2, 657–683.

[27] Stanisław Łojasiewicz, Triangulation of semi-analytic sets. Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, Série 3, Tome 18 (1964) no. 4, pp. 449–474.

[28] Saunders Mac Lane and Ieke Moerdijk, Sheaves in Geometry and Logic: A first introduction to topos theory, Universitext, Springer-Verlag, New York, 1994, Corrected reprint of the 1992 edition.

[29] Barry Mazur, Notes on étale cohomology of number fields, Ann. Sci. École Norm. Sup. (4) **6** (1973), 521–552 (1974).

[30] J. S. Milne, Arithmetic duality theorems, second ed., BookSurge, LLC, Charleston, SC, 2006.

[31] Baptiste Morin, Sur le topos weil-étale d'un corps de nombres, 2008, PhD thesis, Université Bordeaux I.

[32] Baptiste Morin, Zeta functions of regular arithmetic schemes at $s = 0$, Duke Math. J. **163** (2014), no. 7, 1263–1336.

[33] Amnon Neeman, Triangulated categories, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001.

[34] Christian Serpé, Resolution of unbounded complexes in Grothendieck categories, J. Pure Appl. Algebra 177 (2003), no. 1, 103–112.

[35] Christophe Soulé, Groupes de Chow et K-théorie de variétés sur un corps fini, Math. Ann. **268** (1984), no. 3, 317–345.

[36] Nicolas Spaltenstein, Resolutions of unbounded complexes, Compositio Math. 65 (1988), no. 2, 121–154.

[37] Jean-Louis Verdier, Des catégories dérivées des catégories abéliennes, Astérisque (1996), no. 239, xii+253 pp. (1997), With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis.

[38] Bartel L. van der Waerden, Topologische Begründung des Kalküls der abzählenden Geometrie, Math. Ann. **102** (1930), no. 1, 337–362.

[39] Charles A. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

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